

Ques

(1)

$$1. (a) \quad f(x) = x(1+x)^{10}$$

$$\begin{aligned}f'(x) &= (1+x)^{10} + x \times 10(1+x)^9 \\&= (1+x)^9 ((1+x) + 10x) \\&= \underline{(1+x)^9 (1+11x)}\end{aligned}$$

$$(b) \quad y = 3^x$$

$$\ln y = \ln 3^x$$

$$\ln y = x \cdot \ln 3$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln 3 \cdot x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 3$$

$$\Rightarrow \underline{\frac{dy}{dx} = \ln 3 \cdot 3^x}$$

$$2. \quad S_n = 9 + 7 + 5 + \dots$$

arithmetic series. $a = 9, d = -2$

$$\begin{aligned}S_n &= \frac{n}{2} [18 + (n-1) \times -2] = 9n - n(n-1) \\&= 10n - n^2 \\&= 10n - n^2\end{aligned}$$

$$S_n = 21 \Rightarrow 10n - n^2 = 21$$

$$n^2 - 10n + 21 = 0$$

$$(n-7)(n-3) = 0$$

$$n = 7 \text{ or } n = 3$$

(2)

3 $y^3 + 3xy = 3x^2 - 5$

Differentiate

$$3y^2 \cdot \frac{dy}{dx} + 3y + 3x \cdot \frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} [3y^2 + 3x] = 6x - 3y$$

$$\frac{dy}{dx} = \frac{6x - 3y}{3y^2 + 3x} = \frac{3(2x - y)}{3(y^2 + x)}$$

$$\frac{dy}{dx} = \frac{2x - y}{y^2 + x} \quad \text{at } A(2, 1)$$

point $(2, 1)$ $m = 1$

$$m = \frac{4 - 1}{1 + 2} = \underline{\underline{1}}$$

$$y - b = m(x - a)$$

$$y - 1 = x - 2 \Rightarrow y = x - 1$$

4. $|z + i| = 2 \Rightarrow |x + iy + i| = 2$

$$\Rightarrow |x + i(y+1)| = 2$$

$$\Rightarrow \sqrt{x^2 + (y+1)^2} = 2$$

$$\Rightarrow x^2 + (y+1)^2 = 4$$

i.e Circle Centre $(0, -1)$ Radius = 2

$$5. \quad x = 1 + \sin \theta$$

(3)

$$dx = \cos \theta d\theta$$

$$\theta = 0 \quad x = 1 + \sin 0 = 1$$

$$\theta = \frac{\pi}{2} \quad x = 1 + \sin \frac{\pi}{2} = 2$$

$$\text{INT} = \int_1^2 \frac{dx}{x^3} = \int_1^2 x^{-3} dx$$

$$= \left[\frac{x^{-2}}{-2} \right]_1^2 = \left[-\frac{1}{2x^2} \right]_1^2$$

$$= \left(-\frac{1}{8} \right) - \left(-\frac{1}{2} \right) = \underline{\underline{\frac{3}{8}}}$$

$$6. \quad \begin{bmatrix} 1 & 1 & 3 & 1 \\ 3 & a & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array} \quad \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & a-3 & -8 & -2 \\ 0 & 0 & -2 & -2 \end{bmatrix}$$

$$\Rightarrow -2z = -2 \Rightarrow \underline{\underline{z=1}}$$

$$(a-3)y - 8 \times 1 = -2 \Rightarrow (a-3)y = 6$$

$$\Rightarrow \underline{\underline{y = \frac{6}{a-3}}}$$

$$x + y + 3z = 1$$

$$\Rightarrow x + \underline{\underline{\frac{6}{a-3}}} + 3 = 1 \Rightarrow \underline{\underline{x = \frac{-6}{a-3} - 2}}$$

When $a = 3$, the 2nd equation would give $-8z = -2$
 $\Rightarrow z = \frac{1}{4}$

but 3rd equation gives $z=1$ So Equations are inconsistent

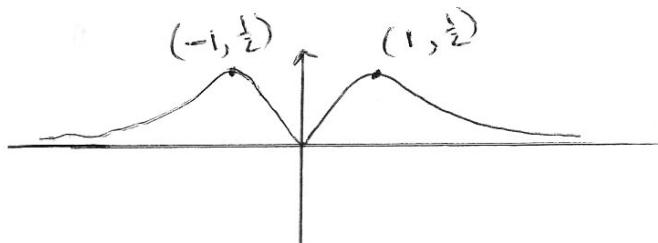
$$7 \quad \frac{dy}{dx} = \frac{1 \cdot (1+x^2) - x \cdot (2x)}{(1+x^2)^2} \quad (4)$$

$$= \frac{1 + x^2 - 2x^2}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} = 0$$

$$\Rightarrow 1-x^2 = 0 \Rightarrow (1-x)(1+x) = 0 \\ \Rightarrow x=1 \text{ or } x=-1$$

$$y = \frac{1}{2} \quad y = -\frac{1}{2}$$

Start Points $(-1, -\frac{1}{2})$ and $(1, \frac{1}{2})$



So 2 stationary points $(-1, \frac{1}{2})$ and $(1, \frac{1}{2})$
and a critical value at $(0, 0)$

(5)

8 A is TRUE

$$p(n) = n^2 + n = n(n+1)$$

Case1 n is even $\Rightarrow n = 2k \quad k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow p(n) &= n(n+1) = 2k(2k+1) \\ &= 2 \times k(2k+1) \end{aligned}$$

$\Rightarrow p(n)$ is even.

Case2 n is odd $\Rightarrow n = 2k+1 \quad k \in \mathbb{N}$

$$\begin{aligned} \Rightarrow p(n) &= n(n+1) = (2k+1)(2k+1+1) \\ &= 2k(2k+1) \\ &= 2 \times k(2k+1) \end{aligned}$$

$\Rightarrow p(n)$ is even

hence $p(n)$ is even for $n \in \mathbb{Z}^+$

B is false

$$n=1 \Rightarrow p(n) = 1+1 = 2$$

and 2 is not a multiple of 3.

(6)

q.

$$\begin{aligned}
 & \frac{1}{\cos\theta + i\sin\theta} \times \frac{(\cos\theta - i\sin\theta)}{(\cos\theta - i\sin\theta)} \\
 &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta} = \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta} \\
 &= \frac{\cos\theta - i\sin\theta}{1} = \frac{\cos\theta - i\sin\theta}{1}
 \end{aligned}$$

$$\omega^k = \cos k\theta + i\sin k\theta$$

$$\omega^{-k} = \frac{1}{\omega^k} = \cos k\theta - i\sin k\theta$$

$$\begin{aligned}
 \omega^k + \omega^{-k} &= \cos k\theta + i\sin k\theta + \cos k\theta - i\sin k\theta \\
 &= \underline{2\cos k\theta}.
 \end{aligned}$$

$$(\omega + \omega^{-1})^4 = (2\cos\theta)^4 = 16\cos^4\theta$$

$$\begin{aligned}
 \text{also } (\omega + \omega^{-1})^4 &= \binom{4}{0}\omega^4 + \binom{4}{1}\omega^3\cdot\omega^{-1} + \binom{4}{2}\omega^2\omega^{-2} \\
 &\quad + \binom{4}{3}\omega\omega^{-3} + \binom{4}{4}\omega^{-4} \\
 &= \omega^4 + 4\omega^2 + 6 + \frac{4}{\omega^2} + \frac{1}{\omega^4} \\
 &= (\omega^4 + \omega^{-4}) + 4(\omega^2 + \omega^{-2}) + 6
 \end{aligned}$$

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$$\Rightarrow 16\cos^4\theta = (\omega^4 + \omega^{-4}) + 4(\omega^2 + \omega^{-2}) + 6$$

$$\Rightarrow 16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$\Rightarrow \cos^4\theta = \frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}$$

10.

$$(a) I_1 = \int_0^1 xe^{-x} dx = \left[-xe^{-x} \right]_0^1 + \int_0^1 e^{-x} dx$$

$$= (-e^{-1}) - (0) + \left[-e^{-x} \right]_0^1$$

$$= -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e}$$

$$(b) I_n = \int_0^1 x^n e^{-x} dx = \left[-x^n e^{-x} \right]_0^1 + \int_0^1 nx^{n-1} e^{-x} dx$$

$$= -e^{-1} + n \int_0^1 x^{n-1} e^{-x} dx$$

$$I_n = n I_{n-1} - e^{-1}$$

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$$(c) I_3 ? \quad I_2 = 2I_1 - e^{-1}$$

$$\Rightarrow I_2 = 2 - \frac{4}{e} - \frac{1}{e} = 2 - \frac{5}{e}$$

$$I_3 = 2I_2 - \frac{1}{e} = 6 - \frac{15}{e} - \frac{1}{e}$$

$$= 6 - \frac{16}{e}$$

11.

$$\frac{dv}{dt} = v(10-v)$$

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$$\Rightarrow \frac{dv}{v(10-v)} = dt$$

$$\Rightarrow \int \frac{dv}{v(10-v)} = \int dt$$



$$\text{Let } \frac{1}{v(10-v)} = \frac{A}{v} + \frac{B}{10-v}$$

$$\Rightarrow 1 = A(10-v) + Bv$$

$$v=0 \Rightarrow 1 = A \times 10$$

$$A = \frac{1}{10}$$

$$v=10 \Rightarrow 1 = 10B$$

$$B = \frac{1}{10}$$

$$\Rightarrow \int \frac{\frac{1}{10}}{v} dv + \int \frac{\frac{1}{10}}{10-v} dv = \int dt$$

$$\Rightarrow \frac{1}{10} \int \frac{dv}{v} + \frac{1}{10} \int \frac{dv}{10-v} = t + C$$

$$\Rightarrow \frac{1}{10} \ln v - \frac{1}{10} \ln(10-v) = t + C \quad \textcircled{1}$$

$$v(0)=5 \Rightarrow t=0 \text{ when } v=5 \text{ fit into } \textcircled{1}$$

$$\frac{1}{10} \ln 5 - \frac{1}{10} \ln 5 = 0 + C \\ \Rightarrow C = 0$$

$$\text{So } \frac{1}{10} (\ln v - \ln(10-v)) = t \Rightarrow$$

$$\ln \left(\frac{v}{10-v} \right) = 10t$$

$$\Rightarrow e^{\ln \left(\frac{v}{10-v} \right)} = e^{10t} \quad (9)$$

$$\Rightarrow \frac{v}{10-v} = e^{10t}$$

$$\Rightarrow v = e^{10t} (10 - v)$$

$$\Rightarrow v = 10e^{10t} - ve^{-10t}$$

$$\Rightarrow v + ve^{10t} = 10e^{10t}$$

$$\Rightarrow v(1 + e^{10t}) = 10e^{10t}$$

$$\Rightarrow v = v(t) = \frac{10e^{10t}}{1 + e^{10t}}$$

as $t \rightarrow \infty$

$$v = \frac{10e^{10t}}{e^{10t} \left(1 + \frac{1}{e^{10t}} \right)} = \frac{10}{1 + \frac{1}{e^{10t}}}$$

now $\frac{1}{e^{10t}} \rightarrow 0$ as $t \rightarrow \infty$

$\therefore 1 + \frac{1}{e^{10t}} \rightarrow 1$ as $t \rightarrow \infty$

$\therefore v(t) \rightarrow \frac{10}{1} = 10$ as $t \rightarrow \infty$.

(12)

(10)

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2} = \lambda$$

$$\Rightarrow x = 4\lambda + 3, y = -\lambda + 2, z = 2\lambda - 1 \quad (1)$$

put these into

$$2x + y - z = 4$$

$$\Rightarrow 8\lambda + 6 - \lambda + 2 - 2\lambda + 1 = 4$$

$$\Rightarrow 5\lambda = -5 \Rightarrow \lambda = -1$$

put $\lambda = -1$ in (1) \rightarrow

$$x = -1, y = 3, z = -3 \quad \text{meets at } (-1, 3, -3)$$

$$13. \quad A^2 = 4A - 3I$$

$$A^4 = A^2 \cdot A^2 = (4A - 3I)(4A - 3I)$$

$$= 16A^2 - 24AI + 9I^2$$

$$= 16(4A - 3I) - 24A + 9I.$$

$$= 64A - 48I - 24A + 9I.$$

$$= \underline{40A - 39I}$$

$$\therefore p = 40, q = -39$$

$$14. \text{ put } x_{n+1} = x_n = f$$

(11)

$$\Rightarrow f = \frac{1}{2}\left(f + \frac{7}{f}\right)$$

$$\Rightarrow 2f = f + \frac{7}{f}$$

$$\Rightarrow f = \frac{7}{f} \Rightarrow f^2 = 7$$

$$f = \sqrt{7} \quad \text{or} \quad f = -\sqrt{7}$$

fixed points are $-\sqrt{7}$ and $\sqrt{7}$.

$$15. f(x) = \sin^2 x \quad f(0) = 0$$

$$f'(x) = 2 \sin x \cos x \quad f'(0) = 0$$

$$= \sin 2x$$

$$f''(x) = 2 \cos 2x \quad f''(0) = 2$$

$$f'''(x) = -4 \sin 2x \quad f'''(0) = 0$$

$$f''''(x) = -8 \cos 2x \quad f''''(0) = -8$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f''''(0)x^4}{4!}$$

$$f(x) = \frac{2x^2}{2!} - \frac{8x^4}{4!} = x^2 - \frac{1}{3}x^4$$

$$\begin{aligned} \cos^2 x &= 1 - \sin^2 x \\ &= 1 - x^2 + \frac{1}{3}x^4 \end{aligned}$$

(12)

$$6(a) \quad n=1$$

$$\text{LHS} = 3(i^2 - i) = 0$$

$$\text{RHS} = (i-1)i(i+1) = 0$$

$$\text{LHS} = \text{RHS}$$

Result is true for $n=1$

Assume result is true for $n=k$

$$\text{i.e. } \sum_{r=1}^k 3(r^2 - r) = (k-1)k(k+1)$$

LHS

$$\text{ADD } 3((k+1)^2 - (k+1))$$

$$\text{LHS} =$$

$$\sum_{r=1}^k 3(r^2 - r) + 3((k+1)^2 - (k+1))$$

$$= \sum_{r=1}^{k+1} 3(r^2 - r) = (k-1)k(k+1) + 3((k+1)^2 - (k+1))$$

$$= (k-1)k(k+1) + 3(k+1)((k+1) - 1)$$

$$= (k-1)k(k+1) + 3k(k+1)$$

$$= k(k+1)[(k-1) + 3]$$

$$= k(k+1)(k+2)$$

$$= ((k+1)-1)(k+1)((k+1)+1)$$

\Rightarrow Result is true for $n=k+1$ whenever true
for $n=k$. Since also true for $n=1$ then

true $\forall n \geq 1$

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(b)

$$\sum_{r=1}^{10} 3(r^2 - r) = 9 \times 10 \times 11 = 990$$

(13)

$$\sum_{r=1}^{40} 3(r^2 - r) = 39 \times 40 \times 41 = 63960$$

$$\sum_{r=1}^{40} \dots - \sum_{r=1}^{10} \dots = \sum_{r=11}^{40} \dots$$

$$= 63960 - 990 = 62970$$

17. Aux Eqn

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

$$\alpha = 2 \text{ (twice)}$$

gen solⁿ is $y = A e^{2x} + B x e^{2x}$
cf.

try PI & $y = C e^x$

$$\frac{dy}{dx} = C e^x \quad \frac{d^2y}{dx^2} = C e^x$$

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = C e^x - 4C e^x + 4C e^x = e^x$$

$$\Rightarrow \underline{C=1}$$

(14)

So gen solⁿ is

$$y = CF + PI$$

$$y = Ae^{2x} + Bxe^{2x} + e^x.$$

Particular Solⁿ

$$y = 2 \text{ when } x = 0$$

$$\rightarrow 2 = A + 1$$

$$A = 1$$

$$\frac{dy}{dx} = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x} + e^x$$

$$\frac{dy}{dx} = 1 \text{ when } x = 0 \rightarrow$$

$$1 = 2 + B + 1 \Rightarrow B = -2$$

$$y = e^{2x} - 2xe^{2x} + e^x$$