

2004

$$\text{Q1 (a)} \quad f(x) = \cos^2 x e^{\tan x}$$

$$f'(x) = 2 \cos x \cdot (-\sin x) \cdot e^{\tan x}$$

$$+ \cos^2 x \cdot e^{\tan x} \cdot \sec^2 x$$

$$= -\sin 2x \cdot e^{\tan x} + \cancel{\cos^2 x} \cdot e^{\tan x} \cdot \frac{1}{\cos^2 x}$$

$$f'(x) = e^{\tan x} (1 - \sin 2x)$$

$$f'(\frac{\pi}{4}) = e^{\tan \frac{\pi}{4}} \left(1 - \sin \frac{\pi}{2}\right) = 0$$

$$(b) \quad g(x) = \frac{\tan^{-1} 2x}{1 + 4x^2}$$

$$g'(x) = \frac{\frac{1}{1+(2x)^2} \cdot 2 \cdot (1+4x^2) - \tan^{-1} 2x \cdot 8x}{(1+4x^2)^2}$$

$$= \frac{2 - 8x \tan^{-1} 2x}{(1+4x^2)^2}$$

$$\begin{aligned}
 2. (a^2 - 3)^4 &= \binom{4}{0} (a^2)^0 (-3)^4 + \binom{4}{1} (a^2)^1 (-3)^3 + \binom{4}{2} (a^2)^2 (-3)^2 \\
 &\quad + \binom{4}{3} (a^2)^3 (-3) + \binom{4}{4} (a^2)^4 (-3)^0 \\
 &= 81 + 4 \times a^2 \times (-27) + 6 a^4 \times 9 \\
 &\quad + 4 \times a^6 \times -3 + a^8 \\
 &= 81 - 108 a^2 + 54 a^4 - 12 a^6 + a^8
 \end{aligned}$$

$$3 \quad \dot{x} = -5 \sin \theta \quad \dot{y} = 5 \cos \theta$$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{5 \cos \theta}{-5 \sin \theta} = -\cot \theta$$

$$\text{When } \theta = \frac{\pi}{4} \quad x = 5 \cos \frac{\pi}{4}, \quad y = 5 \sin \frac{\pi}{4}$$

$$x = \frac{5}{\sqrt{2}}, \quad y = \frac{5}{\sqrt{2}}$$

$$M = \frac{dy}{dx} = -\cot \frac{\pi}{4} = -\frac{1}{\tan \frac{\pi}{4}} = -1$$

point $(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}})$
gradient $= -1$

$$y - b = m(x - a)$$

$$y - \frac{5}{\sqrt{2}} = -1(x - \frac{5}{\sqrt{2}})$$

$$\Rightarrow y - \frac{5}{\sqrt{2}} = -x + \frac{5}{\sqrt{2}}$$

$$\Rightarrow x + y = \frac{10}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow x + y = 5\sqrt{2}$$

$$4. \quad z^2 = (1+2i)(1+2i) = 1 + 4i + (2i)^2 \\ = 1 + 4i - 4$$

$$z+3 = 1+2i+3 = 4+2i = -3+4i$$

$$z^2(z+3) = (-3+4i)(4+2i)$$

$$= -12 + 16i - 6i + 8i^2$$

$$= \underline{-20 + 10i} \quad (1)$$

$$-5z = -5 - 10i$$

$$-5z + 25 = 20 - 10i \quad (2)$$

$$z^3 + 3z^2 - 5z + 25 = z^2(z+3) \\ + (-5z + 25)$$

$$= \overset{(1)}{(-20+10i)} + \overset{(2)}{(20-10i)}$$

$$= 0$$

hence $(1+2i)$ is a root of the equation

$\Rightarrow (1-2i)$ is also a root

$$1+2i \left| \begin{array}{cccc} 1 & 3 & -5 & 25 \end{array} \right.$$

$$\downarrow \quad 1+2i \quad 10i \quad -25$$

$$1-2i \left| \begin{array}{cccc} 1 & 4+2i & -5+10i & 0 \end{array} \right.$$

$$\downarrow \quad 1-2i \quad 5-10i$$

$$\downarrow \quad 1 \quad 5, \quad 0$$

$(z+5)$ factor
 $\Rightarrow z = -5$ Root

Roots are $z = (1+2i), (1-2i)$ and (-5)

$$5. \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

$$\text{Let } \frac{1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$\Rightarrow 1 = A(x+2) + B(x-3) \quad (i)$$

put $x = 3$ in (i)

$$\rightarrow 1 = 5A \Rightarrow A = \frac{1}{5}$$

put $x = -2$ in (i)

$$\rightarrow 1 = -5B \Rightarrow B = -\frac{1}{5}$$

$$= \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$$

$$\int_0^1 \left(\frac{1}{5(x-3)} - \frac{1}{5(x+2)} \right) dx = \frac{1}{5} \int_0^1 \left(\frac{1}{x-3} - \frac{1}{x+2} \right) dx$$

$$= \frac{1}{5} \left[\ln(x-3) - \ln(x+2) \right]_0^1 = \frac{1}{5} \left[\ln \left| \frac{x-3}{x+2} \right| \right]_0^1$$

$$= \frac{1}{5} \ln \left| \frac{-2}{3} \right| - \frac{1}{5} \ln \left| \frac{-3}{2} \right| = \frac{1}{5} \ln \left(\frac{2}{3} \right) - \frac{1}{5} \ln \left(\frac{3}{2} \right)$$

$$= \frac{1}{5} \ln \left(\frac{2}{3} \right) = \frac{1}{5} \ln \left(\frac{4}{9} \right)$$

6.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \downarrow \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \downarrow$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\downarrow \quad \downarrow$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$M_2 M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

So $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

\Rightarrow reflection in line $y = -x$

7. $f(x) = e^x \sin x + e^x \cos x$

$$f'(x) = e^x \sin x + e^x \cos x \quad f(0) = 0$$

$$f''(x) = e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x = 2e^x \cos x \quad f'(0) = 1$$

$$f'''(x) = 2e^x \cos x - 2e^x \sin x \quad f''(0) = 2e^0 \cos 0$$

$$= 2$$

$$f'''(0) = 2 - 0 = 2$$

So $e^x \sin x = f(0) + f'(0) \frac{x}{1!} + f''(0) \cdot \frac{x^2}{2!} + f'''(0) \cdot \frac{x^3}{3!}$

$$= x + \frac{2x^2}{2} + \frac{2x^3}{6} = x + x^2 + \frac{1}{3}x^3$$

8.

$$\text{Line 1} \quad 231 = 13 \times 17 + 10$$

$$\text{Line 2} \quad 17 = 1 \times 10 + 7$$

$$\text{Line 3} \quad 10 = 1 \times 7 + 3$$

$$\text{Line 4} \quad 7 = 2 \times 3 + 1$$

$$\text{Line 5} \quad 3 = 3 \times 1 \longrightarrow (231, 17) = 1$$

Line 4 \rightarrow

$$1 = 2 \times 3 + 1$$

$$\Rightarrow 1 = 7 - 2 \times 3$$

\downarrow Line 3

$$1 = 7 - 2 \times (10 - 1 \times 7)$$

$$\Rightarrow 1 = 3 \times 7 - 2 \times 10$$

\uparrow Line 2

$$\Rightarrow 1 = 3(17 - 1 \times 10) - 2 \times 10$$

$$1 = 3 \times 17 - 5 \times 10$$

\uparrow Line 1

$$1 = 3 \times 17 - 5(231 - 13 \times 17)$$

$$\Rightarrow 1 = 68 \times 17 - 5 \times 231 \quad \text{i.e. } x = -5, y = 68$$

$$9. \quad x = (u-1)^2$$

$$\Rightarrow dx = 2(u-1) du$$

$$(1+\sqrt{x})^3 = (1 + \sqrt{u-1})^3 = (1 + (u-1))^3 = u^3$$

$$\text{INT} = \int \frac{2(u-1) du}{u^3} = \int \left(\frac{2u}{u^3} - \frac{2}{u^3} \right) du$$

$$= \int (2u^{-2} - 2u^{-3}) du$$

$$= \frac{2u^{-1}}{-1} - \frac{2u^{-2}}{-2} + C$$

$$= -\frac{2}{u} + \frac{1}{u^2} + C$$

$$x = (u-1)^2 \Rightarrow (u-1) = \sqrt{x} \Rightarrow u = 1 + \sqrt{x}$$

$$\text{So INT} = \underline{-\frac{2}{(1+\sqrt{x})} + \frac{1}{(1+\sqrt{x})^2} + C}$$

$$10 \quad f(x) = f(-x) \Rightarrow \text{even} \quad f(x) = -f(-x) \Rightarrow \text{odd}$$

$$f(x) = x^4 \sin 2x \quad f(-x) = (-x)^4 \sin(-2x)$$

$\swarrow \quad \searrow$
 $x^4 \cdot -\sin 2x$

$$= -x^4 \sin 2x$$

hence

$$f(-x) = -f(x)$$

hence $f(x) = x^4 \sin 2x$ is an odd function

$$11. \text{ Volume} = \int_0^1 \pi y^2 dx = \pi \int_0^1 y^2$$

$$y^2 = (e^{-2x})^2 = e^{-4x}.$$

$$\text{Volume} = \pi \int_0^1 e^{-4x} dx = \pi \left[\frac{e^{-4x}}{-4} \right]_0^1$$

$$= \pi \left(-\frac{e^{-4}}{4} - \frac{e^0}{-4} \right) = \frac{\pi}{4} \left(1 - \frac{1}{e^4} \right)$$

$$12. \quad n=1$$

$$\text{LHS} = \frac{d}{dx} (xe^x) = e^x + xe^x = (x+1)e^x$$

$$\text{RHS} = (x+1)e^x \quad \text{LHS} = \text{RHS}$$

Result is true for $n=1$

Assume the result is true for $n=k$.

i.e

$$\boxed{\text{Assume } \frac{d^k}{dx^k} (xe^x) = (x+k)e^x}$$

$$\frac{d^{k+1}}{dx^{k+1}} (xe^x) = \frac{d}{dx} \left[(x+k)e^x \right]$$

$$= e^x + (x+k)e^x \quad \text{by Product Rule}$$

$$= e^x [x + (k+1)]$$

So the result is also true for $n=k+1$ whenever it is true for $n=k$. So since also true for $n=1$ then result is true for all $n \geq 1$
by induction

13. (a) $f(x)$ undefined when $x = -2$

So $x = -2$ is a vertical asymptote

$\deg(\text{Num}) = \deg(\text{Den}) \Rightarrow y = k$ is an asymptote

$$\begin{array}{r} 1 \\ x+2 \end{array} \left[\begin{array}{r} x-3 \\ x+2 \\ -5 \end{array} \right] \Rightarrow f(x) = 1 - \frac{5}{x+2}$$

\uparrow
 $y=1$

\Rightarrow

$y = 1$ is an asymptote

$$(b) f(x) = 1 - 5(x+2)^{-1}$$

$$\Rightarrow f'(x) = 5(x+2)^{-2} = \frac{5}{(x+2)^2}$$

and $f'(x) \neq 0$ for any values of x

\Rightarrow no stationary values

$$(c) f''(x) = -10(x+2)^{-3} = \frac{-10}{(x+2)^3}$$

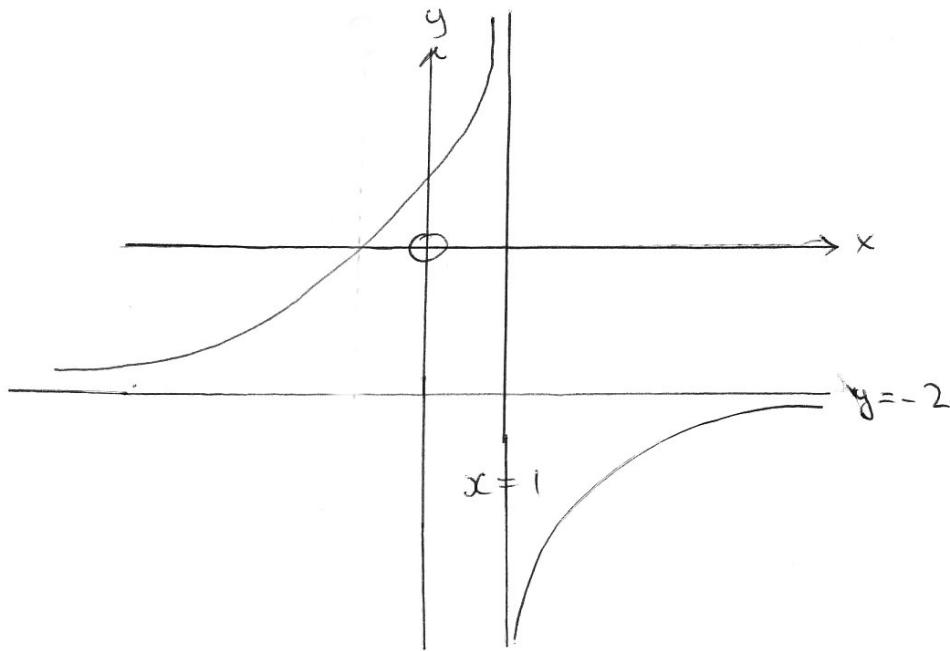
$f''(x) \neq 0$ for any values of x

\Rightarrow no points of inflection

(d) $x = -2$ on $y = f(x)$ becomes $y = -2$ asymptote
on $y = f^{-1}(x)$

$y = 1$ on $y = f(x)$ becomes $x = 1$ asymptote

on $y = f^{-1}(x)$



Domain of $f^{-1}(x)$ $x \in \mathbb{R} : x \neq 1$

$$14(a) \quad \vec{AB} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -4 \end{pmatrix}$$

$$\vec{AC} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}$$

Normal to plane is $\vec{AB} \times \vec{AC}$

$$\begin{array}{r}
 i \quad -1 \quad 0 \quad i \quad -1 \\
 j \quad 2 \quad \times \quad 1 \quad \times \quad 2 \\
 k \quad -4 \quad \times \quad -3 \quad \times \quad -4
 \end{array}
 \quad
 \begin{aligned}
 &= -6i - k + 4i - 3j \\
 &= -2i - 3j - k
 \end{aligned}$$

$$\text{i.e. } \vec{n} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

Equation of plane $\vec{n} \cdot (\vec{r} - \vec{a}) = 0$

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

$$\Rightarrow 2x + 3y + z = 5$$

14 (cont)

angle between planes \Rightarrow angle between normals.

$$\underline{n}_1 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \quad \underline{n}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\cos \theta = \frac{|\underline{n}_1 \cdot \underline{n}_2|}{|\underline{n}_1| \times |\underline{n}_2|}$$

$$\underline{n}_1 \cdot \underline{n}_2 = 2 + 3 - 1 = 4$$

$$|\underline{n}_1| = \sqrt{4+9+1} = \sqrt{14}$$

$$\cos \theta = \frac{4}{(\sqrt{14} \times \sqrt{3})}$$

$$|\underline{n}_2| = \sqrt{1+1+1} = \sqrt{3}$$

$$\theta = 51^\circ 9$$

14(b) line in parametric form

$$x = 4\lambda + 11 \quad y = 5\lambda + 15 \quad z = 2\lambda + 12 \quad (1)$$

fit these into

$$x + y - z = 0$$

$$\Rightarrow 4\lambda + 11 + 5\lambda + 15 - 2\lambda - 12 = 0$$

$$7\lambda = -14 \quad \Rightarrow \underline{\lambda = -2}$$

put in (1)

$$x = -8 + 11 = 3 \quad y = 5 \quad z = -4 + 12 = 8$$

$$\text{ie } (3, 5, 8)$$

$$15(a) \quad \frac{dy}{dx} - \frac{3}{x}y = x^3$$

$$\text{IF} = e^{\int -\frac{3}{x} dx} = e^{-3 \int \frac{1}{x} dx} = e^{-3 \ln x}$$

$$= e^{\ln x^{-3}} = \underline{\underline{x^{-3}}}$$

$$\frac{d}{dx} (\text{IF. } y) = \text{IF. } x^3$$

$$\text{IF. } y = \int \text{IF. } x^3 dx$$

$$\frac{y}{x^3} = \int 1 dx = x + C$$

$$\Rightarrow y = x^4 + xc$$

$$\Rightarrow y = (x+c)x^3$$

$$y=2 \text{ when } x=1 \Rightarrow 2 = (1+c) \cdot 1^3 = 1+c$$

$$\Rightarrow c=1$$

$$\text{Part Sol}^{\text{v}} \text{ is } y = (x+1)x^3$$

$$(b) \quad y \frac{dy}{dx} - 3x = x^4$$

$$y \frac{dy}{dx} = x^4 + 3x$$

$$\Rightarrow \int y dy = \int (x^4 + 3x) dx$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{5}x^5 + \frac{3}{2}x^2 + C$$

$$\Rightarrow y^2 = \frac{2}{5}x^5 + 3x^2 + C$$

$$\Rightarrow y = 2 \text{ when } x=1 \Rightarrow$$

$$(c) \quad A \quad a, d = 2$$

$$B \quad a, r = 2$$

A

$$a + a + d + a + 2d + a + 3d$$

$$= 4a + 6d$$

$$= 4a + 12$$

B

$$a + ar + ar^2 + ar^3$$

$$= a(1 + 2 + 4 + 8)$$

$$= 15a.$$

$$4a + 12 = 15a$$

$$\Rightarrow 11a = 12$$

$$a = \frac{12}{11}$$

=

A

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$= \frac{n}{2} \left[\frac{24}{11} + (n-1) \times 2 \right]$$

$$= \frac{12n}{11} + n^2 - n$$

make equal first

$$\frac{12}{11} (2^n - 1) = 2 \left[\frac{12n}{11} + n^2 - n \right]$$

$$\frac{12}{11} \times 2^n - \frac{12}{11} = \frac{2}{11} n + 2n^2$$

$$\Rightarrow 12 \times 2^n - 12 = 2n + 24n^2$$

try $n = 6$	$LHS = 756$	$RHS = 876$	no not for $n = 6$
try $n = 7$	$LHS = 1524$	$RHS = 1190$	
	$LHS > RHS$	<u>i.e.</u>	$n = 7$ is smallest value

15 (b) cont.

$$4 = \frac{2}{5} + 3 + c$$

$$\Rightarrow c = \frac{3}{5}$$

part soln

$$y = \sqrt{\frac{2}{5}x^5 + 3x^2 + \frac{3}{5}}$$

$$\text{ie } y = \sqrt{2\left(\frac{x^5}{5} + \frac{3x^2}{2} + \frac{3}{10}\right)}$$

$$16 \cdot (a) \quad a = 8, d = 3$$

$$u_n = a + (n-1)d$$

$$8 + (n-1) \times 3 = 56$$

$$\Rightarrow 3n + 5 = 56$$

$$3n = 51$$

$$S_n = \frac{n}{2} [2a + (n-1)d] \quad n = 17$$

$$S_{17} = \frac{17}{2} [16 + 16 \times 3] = 544$$

$$(b) \quad a = 2 \quad a + ar + ar^2 = 266$$

$$\Rightarrow 2 + 2r + 2r^2 = 266$$

$$\Rightarrow r^2 + r + 1 = 133$$

$$\Rightarrow r^2 + r - 132 = 0$$

$$\Rightarrow (r+12)(r-11) = 0$$

$$\Rightarrow r = -12 \quad r = 11$$

not valid
as positive
terms in
sequence