

# Solutions to Advanced Higher 2006

1. Inverse of  $\begin{pmatrix} 2 & x \\ -1 & 3 \end{pmatrix}$

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{6-(-x)} \begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{6+x} \begin{pmatrix} 3 & -x \\ 1 & 2 \end{pmatrix}$$

A matrix is singular if the determinant

$$ad - bc = 0$$

$$\Rightarrow 6+x = 0 \Rightarrow \underline{\underline{x = -6}}$$

2. a)  $2 \tan^{-1} \sqrt{1+x}$

so  $\frac{d}{dx} (2 \tan^{-1} \sqrt{1+x})$

$= 2 \cdot \frac{1}{1+(\sqrt{1+x})^2} \times \frac{1}{2}(1+x)^{-1/2}$

$= \frac{1}{1+1+x} \times \frac{1}{\sqrt{1+x}} = \frac{1}{2+x} \times \frac{1}{\sqrt{1+x}}$

$= \frac{1}{\sqrt{1+x}(2+x)}$

$* \frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$   
x D.O.B.

2.

$$2 b) \quad y = \frac{1 + \ln x}{3x}$$

Quotient Rule

$$\frac{dy}{dx} = \frac{\frac{1}{x}(3x) - 3(1 + \ln x)}{(3x)^2}$$

$$= \frac{3 - 3(1 + \ln x)}{9x^2} = \frac{3 - 3 - 3\ln x}{9x^2}$$

$$= -\frac{3\ln x}{9x^2} = \underline{\underline{-\frac{\ln x}{3x^2}}}$$

$$3. \quad z = -i + \frac{1}{1-i}$$

$$\begin{aligned} \text{change } \frac{1}{(1-i)} \times \frac{(1+i)}{(1+i)} &= \frac{1+i}{1+i-i-i^2} = \frac{1+i}{1-(-1)} = \frac{1+i}{2} \\ &= \frac{1}{2}(1+i) = \underline{\underline{\frac{1}{2} + \frac{1}{2}i}} \end{aligned}$$

$$\text{so } z = -i + \frac{1}{2} + \frac{1}{2}i = \underline{\underline{\frac{1}{2} - \frac{1}{2}i}}$$

$$\text{Mod } z = |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{1}{2}} = \underline{\underline{\frac{1}{\sqrt{2}}}}$$

$$\text{or } \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \underline{\underline{\frac{\sqrt{2}}{2}}}$$

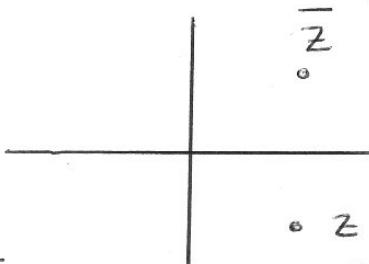
$$\operatorname{Arg} z = \tan^{-1} \left( \frac{-1/2}{1/2} \right)$$

Acute angle =  $45^\circ$

$$z \text{ is in Quad 4} \Rightarrow \operatorname{Arg} z = -\frac{\pi}{4}$$

or  $\frac{7\pi}{4}$

3.



$z = \frac{1}{2} - \frac{1}{2}i$

$\bar{z} = \frac{1}{2} + \frac{1}{2}i$

$$4. xy - xc = 4$$

↑  
Product Rule

$$\underline{1. y + xc \cdot dy/dx - 1 = 0}$$

$$y + xc \cdot dy/dx = 1$$

$$xc \cdot dy/dx = 1 - y$$

$$\underline{\underline{dy/dx = \frac{1-y}{xc}}}$$

$\frac{d^2y}{dx^2}$  use Quotient Rule.

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{dy/dx(x) - 1(1-y)}{xc^2} \\ &= -\frac{xc \cdot dy/dx - 1 + y}{xc^2} \end{aligned}$$

Sub in  $dy/dx$

4.

$$\frac{d^2y}{dx^2} = -\frac{x \left(\frac{1-y}{x}\right) - 1 + y}{x^2}$$

$$= \frac{-1+y-1+y}{x^2} = \underline{\underline{\frac{2y-2}{x^2}}}$$

5. If fixed point is  $L$

$$L = \frac{1}{2} \left( L + \frac{2}{L^2} \right)$$

$$2L = L + \frac{2}{L^2}$$

$$L = \frac{2}{L^2} \Rightarrow L^3 = 2 \Rightarrow \underline{\underline{L = \sqrt[3]{2}}}$$

$$6. \int \frac{12x^3 - 6x}{x^4 - x^2 + 1} dx$$

$$\int \frac{12x^3 - 6x}{u} \cdot \frac{du}{4x^3 - 2x}$$

$$\int \frac{3(4x^3 - 2x)}{u} \cdot \frac{du}{4x^3 - 2x} = \int \frac{3}{u} du = \underline{\underline{3 \ln u + C}}$$

$$\text{sub back} = \underline{\underline{3 \ln(x^4 - x^2 + 1) + C}}$$

$$\text{het } u = x^4 - x^2 + 1$$

$$\frac{du}{dx} = 4x^3 - 2x$$

$$\frac{du}{dx} = \frac{du}{4x^3 - 2x}$$

5.

$$\begin{aligned}
 Q7.a) n^3 - n &= n(n^2 - 1) \\
 &= n(n+1)(n-1) \\
 &= (n+1)(n)(n-1)
 \end{aligned}$$

3 consecutive numbers.

so  $\div 3$  and  $\div 2 \Rightarrow$  divisible by 6.

b)  $n^3 + n + 5$

$$\begin{aligned}
 \text{let } n = 2, n^3 + n + 5 &= 8 + 2 + 5 \\
 &= 15 \quad \text{divisible by } 5/3 \\
 &\quad \text{so not prime.}
 \end{aligned}$$

$$\begin{aligned}
 \text{let } n = 5, n^3 + n + 5 &= 125 + 5 + 5 \\
 &= 135 \quad \text{divisible by 5} \\
 &\quad \text{so not prime.}
 \end{aligned}$$

$\Rightarrow$  Statement is false.

Q8.  $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = 0$  6.

Auxillary Equation

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned} b^2 - 4ac &= 4 - 4(1)(2) \\ &= -4 \\ \Rightarrow &\text{Complex Roots} \end{aligned}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}$$

$$= \underline{-1 \pm i}$$

So solution is  $y = e^{-x} (A \cos x + B \sin x)$

$$\begin{aligned} x &= 0 \\ y &= 0 \\ \frac{dy}{dx} &= 2 \end{aligned}$$

$$* y = e^0 (A \cos 0 + B \sin 0)$$

$$0 = 1 \quad (A) \Rightarrow A = 0$$

$$* \frac{d^2y}{dx^2} = -e^{-x} (A \cos x + B \sin x) + e^{-x} (-A \sin x + B \cos x)$$

$$2 = -1 (0 + B(0)) + 1 (-A(0) + B(1))$$

$$\Rightarrow 2 = B$$

So solution is

$$\underline{y = e^{-x} (2 \sin x)}$$

Q9.  $\left( \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 1 & -2 & 4 & -1 \end{array} \right)$

$$R_3 \rightarrow R_3 - R_2$$

$$\left( \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 1 & 1 & -2 & 2 \\ 0 & -3 & 6 & -3 \end{array} \right)$$

$$R_2 \rightarrow 2R_2 - R_1$$

$$\left( \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -3 & 6 & -3 \end{array} \right)$$

$$\Rightarrow \left( \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & -3 & 6 & -3 \end{array} \right) \rightarrow R_3 \rightarrow R_3 + R_2$$

$$\left( \begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$\Rightarrow z=z \leftarrow$  cannot find value redundant equation.

$$R_2 \rightarrow 3y - 6z = 3$$

$$y = \frac{3+6z}{3} = \underline{\underline{1+2z}}$$

$$R_1 \rightarrow 2x - y + 2z = 1$$

$$2x - (1+2z) + 2z = 1$$

$$2x = 1+1 = 2$$

$$\Rightarrow x = 1$$

$$(1, 1+2z, z)$$

Q16

$$x = T^3 - 90T^2 + 2400T.$$

8.

$$\frac{dx}{dT} = 3T^2 - 180T + 2400$$

$$\Rightarrow 3T^2 - 180T + 2400 = 0$$

$$3(T^2 - 60T + 800) = 0$$

$$3(T - 20)(T - 40) = 0$$

$$\Rightarrow T = 20 \text{ or } T = 40$$

In the Range  $10 \leq T \leq 60$

Find which are max or min

$$\frac{d^2y}{dx^2} = 6T - 180$$

$$\text{at } T = 20 \quad \frac{d^2y}{dx^2} = 120 - 180 = -60 < 0$$

Max Value

$$T = 40 \quad \frac{d^2y}{dx^2} = 240 - 180 = 60 > 0$$

Min Value

$\Rightarrow$  Max removal at  $T = 20^\circ\text{C}$ .

Also check end-points!!!

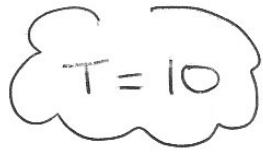
Factors of 8

1, 8

2, 4

?

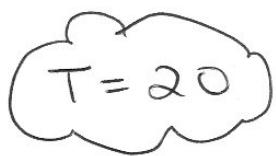
9.

  $T = 10$

$$\begin{aligned}x &= 10^3 - 90(10)^2 + 2400(10) \\&= 1000 - 9000 + 24000 \\&= 16,000\end{aligned}$$

  $T = 60$

$$\begin{aligned}x &= 60^3 - 90(60)^2 + 2400(60) \\&= 36,000\end{aligned}$$

  $T = 20$

$$x = 20,000$$

$\Rightarrow$  Best value at  $T = 60$

$$\text{Q11. } 1 + \cot^2 \theta$$

$$= 1 + \frac{1}{\tan^2 \theta}$$

$$= 1 + \frac{\cos^2 \theta}{\sin^2 \theta}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta}$$

$$= \frac{1}{\sin^2 \theta} = \csc^2 \theta.$$

$$y = \cot^{-1} x \Rightarrow x = \cot y \leftarrow \begin{matrix} \text{Implicit} \\ \text{differentiation} \end{matrix}$$

$$1 = -\csc^2 y \frac{dy}{dx}$$

$$1 = -(1 + \cot^2 y) \frac{dy}{dx}.$$

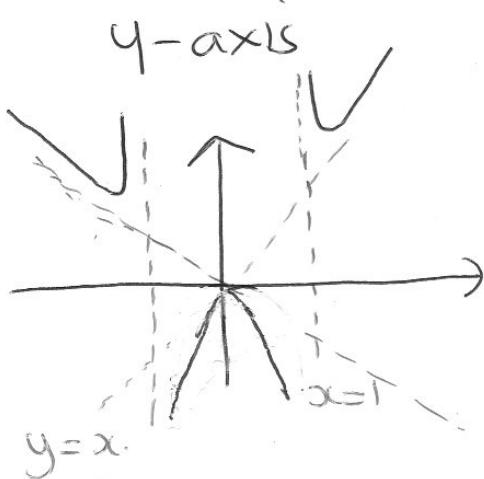
$$\text{so } \frac{dy}{dx} (1 + \cot^2 y) = -1$$

$$\frac{dy}{dx} = \frac{-1}{1 + \cot^2 y}$$

$$\frac{dy}{dx} = -\frac{1}{1+x^2}$$

← sub  
back  
 $x = \cot y$

Q12.  $f$  is an even function  
so it is symmetrical about



other asymptotes  
are  $x = -1$   
 $y = -x$

Q13.  $A^n B = BA^n$

when  $n=1$   $AB = BA \Rightarrow$  True for  $n=1$

Assume true when  $n=k$

$$A^k \cdot B = BA^k$$

Consider  $n=k+1$

$$\begin{aligned}
 \text{L.S } A^{k+1} \cdot B &= A^k \cdot A \cdot B \\
 &= A^k \cdot BA \\
 &= BA^k \cdot A = BA^{k+1}
 \end{aligned}$$

12.

True for  $n=k$

True for  $n=k+1$  hence true  $\forall$  values  
of  $n \geq 1$

13.

$$\text{Q14 a) } f(x) = x^2 \sin x$$

$$f(-x) = (-x)^2 \cdot \sin(-x)$$

<u>s</u>	<u>a</u>
t	c

$$= x^2 \cdot (-\sin x)$$

$$= -x^2 \sin x$$

$$\Rightarrow f(-x) = -f(x) \Rightarrow \text{ODD FUNCTION.}$$

$$(b) \int x^2 \sin x \, dx$$

$$uv - \int v \frac{du}{dx} \, dx$$

Diff		Int
u	$x^2$	$\sin x \frac{dv}{dx}$
$\frac{du}{dx}$	2x	$-\cos x v$

$$= -x^2 \cos x - \int -\cos x (2x) \, dx$$

$$* = -x^2 \cos x + 2 \int x \cos x \, dx$$

$$\text{Let } I = \int x \cos x \, dx$$

$$= x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

Diff		Int
x		$\cos x$
1		$\sin x$

Now put back into \*

$$\int x^2 \sin x \, dx = -x^2 \cos x + 2(x \sin x + \cos x) + C$$

$$\text{14c) Area} = \int_{-\pi/4}^{\pi/4} x^2 \sin x \, dx$$

$$\text{Area} = 2 \times \int_0^{\pi/4} x^2 \sin x \, dx$$

ODD FUNCTION  
IS SYMMETRIC  
ABOUT y-axis

$$\text{Area} = 2 \times \left[ -x^2 \cos x + 2x \sin x + 2 \cos x \right]_0^{\pi/4}$$

$$= 2 \left[ \left( -\left(\frac{\pi}{4}\right)^2 \cos\left(\frac{\pi}{4}\right) + 2\left(\frac{\pi}{4}\right) \sin\left(\frac{\pi}{4}\right) + 2 \cos\left(\frac{\pi}{4}\right) \right) - (0 + 0 + 2 \cos 0) \right]$$

$$= 2 \left[ \left( -\frac{\pi^2}{16} \cdot \left(\frac{1}{\sqrt{2}}\right) + 2\frac{\pi}{4} \left(\frac{1}{\sqrt{2}}\right) + 2\left(\frac{1}{\sqrt{2}}\right) \right) - (2) \right]$$

$$= 2 \left[ \frac{-\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} + \frac{2}{\sqrt{2}} - 2 \right]$$

$$= 2 \left[ -0.43625 + 1.111 + 1.414 - 2 \right]$$

$$= 2 (0.08875)$$

$$= 1.775 \text{ sq units.}$$

Q15. Plane through  $P(1, 1, 0)$

14.

Perpendicular to  $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}$

$\underline{n} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$  ← this is the same as  
the normal

so equation of plane is

$$2x + y - z = 2(1) + (1) - 0$$

$$2x + y - z = 3.$$

Plane and line intersect.

Sub line into plane.

$$x = 2t - 1, y = t + 2, z = -t$$

$$\therefore 2(2t - 1) + (t + 2) - (-t) = 3$$

$$4t - 2 + t + 2 + t = 3$$

$$6t + 0 = 3$$

$$6t = 3 \Rightarrow t = \frac{1}{2}$$

so point  $Q(0, \frac{5}{2}, -\frac{1}{2})$

15.

Shortest distance from P to L

is P to Q

since line is  $\perp$  to plane  
this will give the  
shortest distance

$$P(1, 1, 0) \quad Q\left(0, \frac{5}{2}, -\frac{1}{2}\right)$$

$$\begin{aligned} PQ^2 &= \sqrt{(1-0)^2 + \left(\frac{5}{2}-1\right)^2 + \left(0-\left(-\frac{1}{2}\right)\right)^2} \\ &= \sqrt{1 + \frac{9}{4} + \frac{1}{4}} = \sqrt{\frac{14}{4}} = \frac{\sqrt{14}}{2}. \\ &= 1.87 \text{ units.} \end{aligned}$$

16. a) Common ratio

$$\frac{(x+1)}{(x-2)}$$

$$\begin{aligned} \text{nth term } U_n &= ar^{n-1} \\ &= \frac{x(x+1)}{x-2} \left(\frac{x+1}{x-2}\right)^{n-1} \\ &= x \left(\frac{x+1}{x-2}\right)^n \end{aligned}$$

$$(b) S_n = \frac{a(r^n - 1)}{r - 1}$$

16.

$$S_n = \frac{x(x+1)}{x-2} \left( \left( \frac{x+1}{x-2} \right)^n - 1 \right)$$

$$= \frac{x(x+1)^{n+1}}{(x-2)^{n+1}} - \frac{x(x+1)}{x-2}$$

$$= \frac{x(x+1)^{n+1}}{(x-2)^{n+1}} - \frac{x(x+1)}{x-2}$$

$$= \frac{1}{3} \left[ \frac{x(x+1)^{n+1}}{(x-2)^n} - x(x+1) \right]$$

$$= \frac{1}{3} x(x+1) \left[ \frac{(x+1)^n}{(x-2)^n} - 1 \right]$$

c) Range for sum to infinity.

17.

$$-1 < r < 1$$

$$-1 < \frac{x+1}{x-2} < 1$$

$$\Rightarrow \text{square up so } \left(\frac{x+1}{x-2}\right)^2 < 1$$

$$x^2 + 2x + 1 < x^2 - 4x + 4$$

$$6x < 3$$

$$x < \frac{1}{2}$$

$$S_\infty = \frac{a}{1-r} = \frac{x\left(\frac{x+1}{x-2}\right)}{1 - \frac{x+1}{x-2}}$$

$$= \frac{x\left(\frac{x+1}{x-2}\right)}{\frac{x-2 - (x+1)}{x-2}}$$

$$= \frac{x\left(\frac{x+1}{x-2}\right)}{\frac{-3}{x-2}} = x(x-2)\left(\frac{x+1}{x-2}\right)$$

$$= -\frac{1}{3}x(x+1)$$

Q17 a)  $\int \sin^2 x \cos^2 x dx$

$\xrightarrow{\quad}$

Change to  $\cos x$   
 $\sin^2 x = 1 - \cos^2 x$

$$= \int (1 - \cos^2 x) \cos^2 x dx$$

$$= \int \cos^2 x - \cos^4 x dx = \int \cos^2 x dx - \int \cos^4 x dx$$

b)  $\int_0^{\pi/4} \cos^4 x dx = \int_0^{\pi/4} \cos x \cdot \cos^3 x dx$

	Diff	Int
$u \rightarrow (\cos x)^3$		$\cos x \leftarrow \frac{dv}{dx}$
$\frac{du}{dx}$	$3 \cos^2 x (-\sin x)$ $= -3 \sin x \cos^2 x$	$\sin x \leftarrow v$

$$uv - \int v \frac{du}{dx} dx$$

$$= \cos^3 x \cdot \sin x - \int \sin x \cdot (-3 \sin x \cos^2 x) dx$$

$$= \cos^3 x \sin x + 3 \int \sin^2 x \cos^2 x dx$$

$$= \left[ \cos^3 x \cdot \sin x \right]_0^{\pi/4} + 3 \int_0^{\pi/4} \sin^2 x \cos^2 x dx$$

19.

$$= \left[ \left( \cos \frac{\pi}{4} \right)^3 \cdot \sin \left( \frac{\pi}{4} \right) - \left( \cos^3 0 \cdot \sin 0 \right) \right] + 3 \int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x dx$$

$$= \left( \left( \frac{1}{\sqrt{2}} \right)^3 \cdot \frac{1}{\sqrt{2}} \right) - 0 + 3 \int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x dx$$

$$= \frac{1}{4} + 3 \int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x dx .$$

$$\begin{aligned} & \int \sin^2 x \cos^2 x \\ &= \int \cos^2 x - \int \cos^4 x . \end{aligned}$$

$$(c) \int_0^{\frac{\pi}{4}} \cos^2 x = ?$$

$$\text{From } \int_0^{\frac{\pi}{4}} \cos^4 x dx = \frac{1}{4} + 3 \int_0^{\frac{\pi}{4}} \sin^2 x \cos^2 x dx$$

$$\int_0^{\frac{\pi}{4}} \cos^4 x dx = \frac{1}{4} + 3 \left[ \int_0^{\frac{\pi}{4}} \cos^2 x dx - \int_0^{\frac{\pi}{4}} \cos^4 x dx \right]$$

$$\int_0^{\frac{\pi}{4}} \cos^4 x dx = \frac{1}{4} + 3 \int_0^{\frac{\pi}{4}} \cos^2 x dx - 3 \int_0^{\frac{\pi}{4}} \cos^4 x dx$$

$$\Rightarrow 4 \int_0^{\frac{\pi}{4}} \cos^4 x dx = \frac{1}{4} + 3 \int_0^{\frac{\pi}{4}} \cos^2 x dx \quad \nwarrow \text{ Given this}$$

$$= \frac{\pi+2}{8}$$

$$\Rightarrow 4 \int_0^{\frac{\pi}{4}} \cos^4 x dx = \frac{1}{4} + 3 \left( \frac{\pi+2}{8} \right)$$

$$4 \int_0^{\pi/4} \cos^4 x dx = \frac{1}{4} + \frac{3\pi}{8} + \frac{3}{4}$$

$$= 1 + \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\pi/4} \cos^4 x = \frac{1 + \frac{3\pi}{8}}{4}$$

$$= \frac{1}{4} + \frac{3\pi}{32}$$

$$= \frac{8 + 3\pi}{32}$$