

## Solution 2007

$$1. \left(x - \frac{2}{x}\right)^4$$



$$\begin{aligned}
 &= 1 \cdot (x)^4 + 4(x)^3\left(-\frac{2}{x}\right) + 6(x)^2\left(-\frac{2}{x}\right)^2 + 4(x)\left(-\frac{2}{x}\right)^3 \\
 &\quad + 1\left(-\frac{2}{x}\right)^4 \\
 &= x^4 - 8x^2 + 24 - \frac{32}{x^2} + \frac{16}{x^4}
 \end{aligned}$$

$$2 \text{ a) } f(x) = \exp(\sin 2x)$$

$$f'(x) = \exp(\sin 2x) \times 2 \cos 2x$$

$$\text{b) } y = 4^{x^2+1}$$

Logarithmic Differentiation

$$\ln y = (x^2+1) \ln 4$$

$$\frac{1}{y} \frac{dy}{dx} = 2x \cdot \ln 4$$

$$\frac{dy}{dx} = 2x y \ln 4$$

$$= \frac{2x 4^{x^2+1} \cdot \ln 4}{}$$

$$3. \quad 3+3i \quad \left| \begin{array}{cccc} 1 & 0 & -18 & 108 \\ \downarrow & 3+3i & 18i & -108 \\ 1 & 3+3i & -18+18i & 0 \end{array} \right.$$

$\Rightarrow$  Remainder = 0  
 $\Rightarrow 3+3i$  is a root.

other root is the conjugate

3-3i

$$\begin{array}{c|ccc} 3-3i & 1 & 3+3i & -18+18i \\ & \downarrow & 3-3i & 18-18i \\ \hline & 1 & 6 & 0 \end{array}$$



$$z+6=0 \Rightarrow z=-6$$

Roots  $3 \pm 3i, -6$

$$Q4. \frac{2x^2-9x-6}{x(x^2-x-6)} = \frac{2x^2-9x-6}{x(x+2)(x-3)}$$

$$\text{so let } \frac{2x^2-9x-6}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$\Rightarrow \{ 2x^2-9x-6 = A(x+2)(x-3) + Bx(x+2) + Cx(x-3)$$

$$\underline{\text{let } x=3} \quad 18-27-6 = 15B \quad \Rightarrow B = -1$$

$$\underline{-15 = 15B}$$

$$\underline{\text{let } x=-2} \quad 8+18-6 = 10C \quad \Rightarrow C = 2$$

$$\underline{20 = 10C}$$

$$\underline{\text{let } x=0} \quad -6 = -6A \quad \Rightarrow A = 1$$

$$\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} = \frac{1}{x} - \frac{1}{(x-3)} + \frac{2}{(x+2)}$$

$$\Rightarrow \int_4^6 \frac{1}{x} - \frac{1}{(x-3)} + \frac{2}{(x+2)} dx$$

$$\Rightarrow \left[ \ln x - \ln(x-3) + 2 \ln(x+2) \right]_4^6$$

$$= (\ln 6 - \ln 3 + 2 \ln 8) - (\ln 4 - \ln 1 + 2 \ln 6)$$

$$= \ln 6 - \ln 3 + \ln 8^2 - \ln 4 + \ln 1 - \ln 6^2$$

$$= \ln \frac{(6 \times 64 \times 1)}{(3 \times 4 \times 36)} = \ln \frac{8}{9}$$

$$\Rightarrow \underline{\underline{m=8, n=9}}$$

$$Q5 \text{ a) } AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$

b) Det A

$$\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{array} = (2) - (-1) = \underline{\underline{3}}$$

Det AB

$$\begin{array}{cccc|cccc} x & x & x & x & x & x \\ -6 & 6 & -1 & -6 & 6 & -1 \\ 0 & 0 & 8 & 0 & 0 & 8 \end{array} \quad 48x - (-48x) \\ = \underline{\underline{96x}}$$

$$\text{Det } A \times \text{Det } B = \text{Det } (AB)$$

$$3 \times \text{Det } B = 96x$$

$$\text{Det } B = \frac{96x}{3} = \underline{\underline{32x}}$$

Q6.  $f(x) = \cos x$        $f(0) = 1$   
 $f'(x) = -\sin x$        $f'(0) = 0$   
 $f''(x) = -\cos x$        $f''(0) = -1$   
 $f'''(x) = \sin x$        $f'''(0) = 0$   
 $f^{IV}(x) = \cos x$        $f^{IV}(0) = 1$

$$f(x) = \frac{f(0)}{0!} + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{IV}(0)x^4}{4!}$$

$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

Deduce  $f(x) = \frac{1}{2} \cos(2x)$ . we know  $\cos x$ , so half  
and sub  $2x$  for  $x$ .

$$\begin{aligned}\frac{1}{2} \cos(2x) &= \frac{1}{2} \left( 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{24} + \dots \right) \\ &= \frac{1}{2} - \frac{x^2}{2} + \frac{1}{3} x^4 + \dots\end{aligned}$$

$$f(x) = \frac{1}{2} \cos(x) = \frac{1}{2} - x^2 + \frac{1}{3} x^4 + \dots$$

$$\begin{aligned} f(3x) &= \frac{1}{2} \cos(3x) = \frac{1}{2} - (3x)^2 + \frac{1}{3} (3x)^4 + \dots \\ &= \frac{1}{2} - 9x^2 + 27x^4 + \dots \end{aligned}$$

Q7.  $\boxed{599p + 53q = 1}$

$$\begin{array}{rcl} 599 & = & 11 \times \boxed{53} + \boxed{16} \\ 53 & = & 3 \times \boxed{16} + \boxed{5} \\ 16 & = & 3 \times \boxed{5} + \boxed{1} \\ 5 & = & 5 \times \boxed{1} + 0 \end{array}$$

sub back for all remainders , 2nd last line

$$\begin{aligned} 1 &= 16 - 3(5) \\ 1 &= 16 - 3[53 - 3(16)] \quad \text{sub for } 5 \\ &= 16 - 3(53) + 9(16) \\ &= 10(16) - 3(53) \leftarrow \text{sub for } 16 \\ &= 10(599 - 11(53)) - 3(53) \\ &= 10(599) - 110(53) - 3(53) \\ &= 10(599) - 113(53) \end{aligned}$$

$\Rightarrow p = 10 \quad q = -113$

$$Q8. \quad \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$$

Auxilliary Equation  $\alpha^2 + 6\alpha + 9 = 0$   
 $(\alpha + 3)(\alpha + 3) = 0$   
 $\Rightarrow \alpha = -3$

Equal roots  $y = A e^{-3x} + Bx e^{-3x}$

P.I Try  $y = C e^{2x}$   
 $\frac{dy}{dx} = 2C e^{2x}$   
 $\frac{d^2y}{dx^2} = 4C e^{2x}$

Now sub into first equation

$$4C e^{2x} + 6(2C e^{2x}) + 9(C e^{2x}) = e^{2x}$$

$$4C + 12C + 9C = 1 \\ \Rightarrow 25C = 1 \Rightarrow C = \frac{1}{25}$$

so P.I  $y = \frac{1}{25} e^{2x}$

Solution  $y = A e^{-3x} + Bx e^{-3x} + \frac{1}{25} e^{2x}$

Q9. Sum of n terms.

$$\sum_{r=1}^n (4 - 6r) = n - 3n^2$$

Find some terms.

$$\begin{aligned} r=1 \quad 4 - 6(1) &= -2 \\ r=2 \quad 4 - 6(2) &= -8 \\ r=3 \quad 4 - 6(3) &= -12 \end{aligned} \quad \left. \right\}$$

$$so \quad a = -2 \quad d = -4$$

$$\begin{aligned} S_n &= \frac{n}{2} \{2a + (n-1)d\} = \frac{n}{2} \{-4 + (n-1)-6\} \\ &= \frac{n}{2} \{-4 - 6n + 6\} = \frac{n}{2} \{-6n + 2\} = \underline{\underline{-3n^2 + n}} \end{aligned}$$

$$\sum_{r=1}^{2q} (4 - 6r) \text{ replace } n \text{ by } 2q$$

$$\Rightarrow \sum_{r=1}^{2q} (4 - 6r) = -3(2q)^2 + 2q = \underline{\underline{-12q^2 + 2q}}$$

$$\begin{aligned} \sum_{r=q+1}^{2q} (4 - 6r) &= \sum_{r=1}^{2q} \dots - \sum_{r=1}^q \\ &= (-12q^2 + 2q) - (-3q^2 + q) \\ &= \underline{\underline{-9q^2 + q}} \end{aligned}$$

$$Q10 \int_0^1 \frac{x^3}{(1+x^2)^4} dx \quad \text{let } u = 1+x^2$$

$$= \int \frac{x^3}{u^4} \cdot \frac{du}{2x} = \int \frac{1}{2} \frac{x^2}{u^4} du$$

$$\frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

need to change to u's

$$= \frac{1}{2} \int \frac{u-1}{u^4} du$$

$$u = 1+x^2 \Rightarrow x^2 = u-1$$

$$= \frac{1}{2} \int \frac{1}{u^3} - \frac{1}{u^4} du = \frac{1}{2} \left[ \frac{u^{-2}}{-2} - \frac{u^{-3}}{-3} \right]_*$$

$$= \frac{1}{2} \left[ -\frac{1}{2u^2} + \frac{1}{3u^3} \right]_*$$

\* New limits.

$$\text{when } x=1, u=2 \\ x=0, u=1$$

$$= \frac{1}{2} \left[ \left( -\frac{1}{2(2)^2} + \frac{1}{3(2)^3} \right) - \left( -\frac{1}{2(1)^2} + \frac{1}{3(1)^3} \right) \right]$$

$$= \frac{1}{2} \left[ -\frac{1}{8} + \frac{1}{24} + \frac{1}{2} - \frac{1}{3} \right] = \frac{1}{2} \left[ -\frac{3}{24} + \frac{1}{24} + \frac{12}{24} - \frac{8}{24} \right]$$

$$= \frac{1}{2} \left( \frac{2}{24} \right) = \underline{\underline{\frac{1}{24}}}$$

Volume of revolution about x-axis

$$V = \int \pi y^2 dx = \pi \int_0^1 \left( \frac{x^{3/2}}{(1+x^2)^2} \right)^2 dx$$

$$= \pi \int \frac{x^3}{(1+x^2)^4} dx \quad \leftarrow \text{same as above}$$

$$= \pi \left[ \frac{1}{24} \right] = \underline{\underline{\frac{\pi}{24}}}$$

Q11.  $|z-2| = |z+i|$       *let  $z = x+iy$*

$$|x+iy-2| = |x+iy+i|$$

$$|(x-2)+iy| = |x+(y+1)i|$$

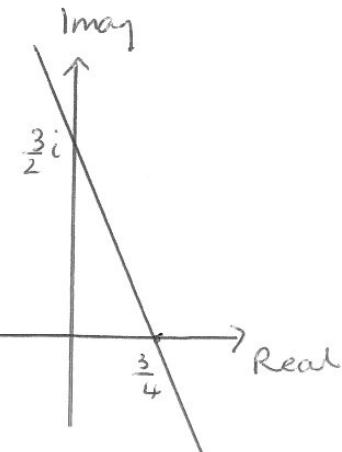
$$\sqrt{(x-2)^2 + y^2} = \sqrt{x^2 + (y+1)^2}$$

$$(x-2)^2 + y^2 = x^2 + (y+1)^2$$

$$x^2 - 4x + 4 + y^2 = x^2 + y^2 + 2y + 1$$

$$-4x + 4 = 2y + 1$$

$$\underline{4x + 2y - 3 = 0}$$



Q12  $(1+a)^n \geq 1+na$

when  $n=1$   $(1+a) \geq 1+a \Rightarrow$  True when  $n=1$

Assume true when  $n=k$

$$(1+a)^k \geq 1+ka *$$

Consider  $n=k+1$  ie the next one

$$(1+a)^{k+1} = (1+a)^k(1+a) \quad \text{we know from *}$$

$$\geq^* (1+ka)(1+a)$$

$$(1+a+ka+ka^2)$$

$$(1+a(k+1)+ka^2)$$

$\hookrightarrow$  this must be  $\geq 0$

$\Rightarrow$  True when  $n=1$ , assume true  $n=k$ , true  $n=k+1$   
True  $\forall n \geq 1$

$$Q13 \quad x = \cos 2t \quad y = \sin 2t$$

$$\text{a) } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\left. \begin{array}{l} \frac{dx}{dt} = -\sin 2t \times 2 \\ \frac{dy}{dt} = \cos 2t \times 2 \end{array} \right\} \quad \frac{dy}{dx} = -\frac{\cos 2t}{\sin 2t}$$

Equation of tangent when  $t = \pi/8$

- Need - gradient,  $\frac{dy}{dx}$   
- a point  $x = ?$ ,  $y = ?$

Gradient  $\frac{dy}{dx} = -\frac{\cos 2(\pi/8)}{\sin 2(\pi/8)} = -1$

Point  $x = \cos 2(\pi/8) = \frac{1}{\sqrt{2}}$

$$y = \sin 2(\pi/8) = \frac{1}{\sqrt{2}} \quad \underline{\left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)}$$

Equation

$$y - b = m(x - a)$$

$$y - \frac{1}{\sqrt{2}} = -1 \left( x - \frac{1}{\sqrt{2}} \right)$$

$$y = -x + \frac{2}{\sqrt{2}}, \quad \underline{\underline{y = -x + \sqrt{2}}}$$

13 b)  $\frac{d^2y}{dx^2} = \frac{d/dt \left( \frac{dy}{dx} \right)}{dx/dt}$  i.e derivative of derivative divided by  $dx/dt$

$$= \frac{d}{dt} \left[ \frac{-\cos 2t}{\sin 2t} \right]$$

$$= \frac{2 \sin 2t (\sin 2t) - 2 \cos 2t (-\cos 2t)}{\sin^2 2t}$$

$$(-2 \sin 2t)$$

$$= \frac{2 \sin^2 2t + 2 \cos^2 2t}{-2 \sin^3 2t} = \frac{2(\sin^2 2t + \cos^2 2t)}{-2 \sin^3 2t}$$

$$= -\frac{1}{\sin^3 2t}$$

Now show

$$\sin 2t \left( \frac{d^2y}{dx^2} \right) + \left( \frac{dy}{dx} \right)^2 = k$$

$$L.S. = \sin 2t \left( -\frac{1}{\sin^3 2t} \right) + \left( -\frac{\cos 2t}{\sin 2t} \right)^2$$

$$= -\frac{1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t}$$

$$= \frac{\cos^2 2t - 1}{\sin^2 2t} = \frac{1 - \sin^2 2t - 1}{\sin^2 2t}$$

$$= -\frac{\sin^2 2t}{\sin^2 2t} = -1$$

$$\Rightarrow \underline{k = -1}$$

$$\text{Q14} \quad \frac{dq}{dt} = \frac{25k - q}{25}$$

$$\int \frac{1}{25k - q} dq = \int \frac{1}{25} dt$$

$$\Rightarrow -\ln(25k - q) = \frac{1}{25} t + C$$

a)  $q=0$  when  $t=0$

$$\Rightarrow -\ln(25k) = C$$

$$\Rightarrow -\ln(25k - q) = \frac{1}{25} t - \ln(25k)$$

$$= -\frac{1}{\sin^2 2t} + \frac{\cos^2 2t}{\sin^2 2t}$$

$$= \frac{\cos^2 2t - 1}{\sin^2 2t} = \frac{1 - \sin^2 2t - 1}{\sin^2 2t}$$

$$= -\frac{\sin^2 2t}{\sin^2 2t} = -1$$

$$\Rightarrow \underline{k = -1}$$

$$\text{Q14} \quad \frac{dq}{dt} = \frac{25k - q}{25}$$

$$\int \frac{1}{25k - q} dq = \int \frac{1}{25} dt$$

$$\Rightarrow -\ln(25k - q) = \frac{1}{25} t + C$$

a)  $q=0$  when  $t=0$

$$\Rightarrow -\ln(25k) = C$$

$$\Rightarrow -\ln(25k - q) = \frac{1}{25} t - \ln(25k)$$

Multiply by -1

$$\ln(25k - q) = -\frac{t}{25} + \ln 25k$$

$$\ln(25k - q) = \ln e^{-t/25} + \ln 25k$$

↑  
need to change to ln  
to combine

$$\ln(25k - q) = \ln e^{-t/25} \cdot 25k$$

$$25k - q = e^{-t/25} \cdot 25k$$

$$\begin{aligned}q &= 25k - e^{-t/25} \cdot 25k \\&= 25k(1 - e^{-t/25})\end{aligned}$$

b)  $q = 0.6$

$t = 5$ .

$k = ?$

$$0.6 = 25k(1 - e^{-5/25})$$

$$0.6 = 25k(1 - 0.8187)$$

$$0.6 = 25k(0.18126)$$

$k = 0.132$

$$(c) \quad b = 10 \\ q = 1$$

$$G = 25k(1 - e^{-t/25})$$

$$= 25(0.132) \left[ 1 - e^{-\frac{10}{25}} \right]$$

$$= 3.3 \left[ 1 - 0.6703 \right]$$

$$= \underline{\underline{1.08801 \text{m}}}$$

So claim is true.

d) Long term height if initial height is 0.3m.

Increase in height,  $q$

$$G = 3.3 \left( 1 - e^{-t/25} \right)$$

check large values of  $t$  i.e  $t = 100, t = 1000$

$t = 10,000$

$$q \approx 3.3(1-0)$$

$$\approx 3.3$$

So tree will grow about 3.3m giving a height of approx 3.6m

Q15. Equate x's and y's solve  
simultaneous equations check z's.

$$L_1 \quad x = 2 + s$$

$$L_2 \quad x = -1 - 2t$$

$$\Rightarrow 2 + s = -1 - 2t$$

$$\Rightarrow \boxed{s + 2t = -3} \quad \text{--- } ①$$

$$L_1 \quad y = -s$$

$$L_2 \quad y = t$$

$$\Rightarrow -s = t$$

$$\boxed{s + t = 0} \quad \text{--- } ②$$

Solve

$$\begin{array}{r} s + 2t = -3 \\ s + t = 0 \\ \hline \end{array}$$

$$t = -3$$

sub  $\boxed{t = -3 \Rightarrow s = 3}$

Now check to see if z coords are the same:  
if yes  $\Rightarrow$  intersect  
if no  $\Rightarrow$  No intersection

$$\begin{aligned}L_1 \quad z &= 2 - s \\&= 2 - 3 \\&= -1\end{aligned}$$

$$\begin{aligned}L_2 \quad z &= 2 + 3t \\&= 2 + (-9) \\&= -7\end{aligned}$$

$\Rightarrow$  Do not intersect.

as point on  $L_1$  is  $(5, -3, -1)$   
on  $L_2$  is  $(5, -3, -7)$

(b)  $L_3$  passes through  $(1, 1, 3)$

Direction perp to directions of both  $L_1 + L_2$

so cross  $L_1 + L_2$  directions to get  $L_3$

$$\begin{array}{ccccccccc} i & 1 & -2 & i & 1 & -2 \\ j & -1 & 1 & j & -1 & 1 \\ k & -1 & 3 & k & -1 & 3 \end{array}$$

$$\begin{aligned}&= (-3i + k + 2j) - (2k - i + 3j) \\&= -2i - j - k = \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \text{ or } \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

Equation of line 3

point  $(1, 1, 3)$        $\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$$\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-3}{1}$$

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(c) Point Q where  $L_3 + L_2$  intersect

$$L_3 : x = 2\lambda + 1, y = \lambda + 1, z = \lambda + 3.$$

$$L_2 : x = -1 - 2t, y = t, z = 2 + 3t$$

$$\Rightarrow x's \quad 2\lambda + 1 = -1 - 2t \quad \Rightarrow \boxed{2\lambda + 2t = -2} \text{ -①}$$

$$\Rightarrow y's \quad \lambda + 1 = t \quad \Rightarrow \boxed{\lambda - t = -1} \text{ -②}$$

Solve simult Equ

$$\begin{array}{r} 2\lambda + 2t = -2 \\ 2\lambda - 2t = -2 \\ \hline 4\lambda = -4 \quad \Rightarrow \lambda = -1 \end{array}$$

sub to find t into ②

$$\begin{aligned} \lambda - t &= -1 \\ -1 - t &= -1 \\ \Rightarrow t &= 0 \end{aligned}$$

when  $\lambda = -1$ ,  $t = 0$

$$x = -1, y = 0$$

$z$ 's

$$\begin{aligned} z &= \lambda + 3 & \text{or} & \quad z = 2 + 3t \\ &= -1 + 3 & & = 2 + 0 \\ &= 2 & & = 2 \end{aligned}$$

$\Rightarrow$  Intersect at point  $Q(-1, 0, 2)$

Lies on line  $L_1$ ,  $L_1: x = 2 + s \Rightarrow 1 = 2 + s, s = -1$   
 $P(1, 1, 3) \quad y = -s \Rightarrow 1 = -s, s = -1$   
 $z = 2 - s \Rightarrow 3 = 2 - s, s = -1$

so  $P$  lies on  $L_1$

d) Shortest distance  $P(1, 1, 3)$  and  $Q(-1, 0, 2)$

$$\begin{aligned} PQ^2 &= \sqrt{(1 - (-1))^2 + (1 - 0)^2 + (3 - 2)^2} \\ &= \sqrt{4 + 1 + 1} = \sqrt{6}. \end{aligned}$$

$$Q16 \cdot a) \tan^{-1}(2x)$$

asymptotes  $y = \pm \frac{\pi}{2}$

$$b) \text{Area} = \int_0^{\frac{1}{2}} \tan^{-1} 2x \, dx$$

use dummy integral

$$= \int 1 \cdot \tan^{-1} 2x \, dx$$

Diff	Int
$\tan^{-1} 2x$	1
$\frac{1}{1+(2x)^2} \cdot 2$	$x$

$$= x \tan^{-1}(2x) - \int \frac{2x}{1+4x^2} \, dx$$

$$\text{let } u = 1+4x^2$$

$$= x \tan^{-1}(2x) - \int \frac{2x}{u} \cdot \frac{du}{8x} \quad \frac{du}{dx} = 8x$$

$$= x \tan^{-1}(2x) - \int \frac{1}{u} \cdot \frac{1}{4} \, du$$

$$= x \tan^{-1}(2x) - \frac{1}{4} \ln u \Big|_{\frac{1}{2}}$$

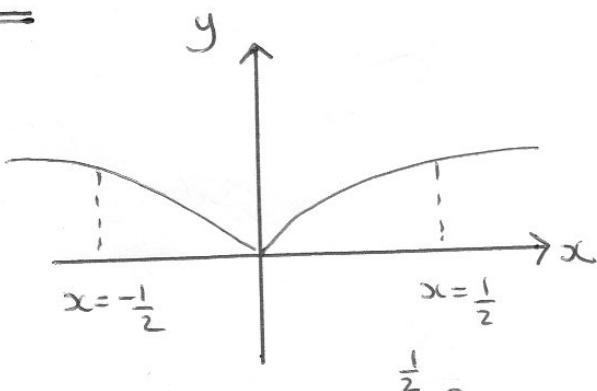
$$= \left[ x \tan^{-1}(2x) - \frac{1}{4} \ln(1+4x^2) \right]_0^{\frac{1}{2}}$$

$$= \left[ \frac{1}{2} \tan^{-1} 1 - \frac{1}{4} \ln 2 \right] - \left[ 0 - \frac{1}{4} \ln 1 \right]$$

$$= \frac{1}{2} \left( \frac{\pi}{4} \right) - \frac{1}{4} \ln 2 + \frac{1}{4} \ln 1$$

$$= \frac{\pi}{8} - \frac{1}{4} \ln 2$$

(c)  $y = |f(x)|$



symmetrical so Area =  $2 \times \int_0^{\frac{1}{2}} \tan^{-1}(2x) dx.$

= 2 × previous answer

$$= 2 \left[ \frac{\pi}{8} - \frac{1}{4} \ln 2 \right]$$

$$= \frac{\pi}{4} - \frac{1}{2} \ln 2$$