

$$1 \text{ (a)} \quad f'(x) = e^x \sin x^2 + e^x \cdot \cos x^2 \cdot 2x \\ = e^x (\sin x^2 + 2x \cos x^2)$$

$$\text{(b)} \quad g'(x) = \frac{3x^2(1+\tan x) - x^3 \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{x^2(3 + 3\tan x - x \sec^2 x)}{(1+\tan x)^2}$$

2.

$$a, ar, ar^2, \dots$$

$$r = \frac{ar^2}{ar} = \frac{3}{-6} = -\frac{1}{2}$$

$$ar = -6, ar^2 = 3$$

$$\text{so } ar = -6 \Rightarrow a = 12$$

as $-1 < r < 1$, Series has a

sum to infinity

$$S_{\infty} = \frac{a}{1-r} = \frac{12}{1 + \frac{1}{2}} = \frac{12}{\frac{3}{2}} = 12 \times \frac{2}{3} \\ = 8$$

$$3(a) \quad t = x^4 \quad I = \frac{1}{4} \int \frac{dt}{1+t^2} = \frac{1}{4} \tan^{-1} t + C$$

$$dt = 4x^3 dx \quad \frac{1}{4} dt = x^3 dx$$

$$= \frac{1}{4} \tan^{-1}(x^4) + C$$

$$(b) \text{ IP. } \int x^2 \ln x \, dx$$

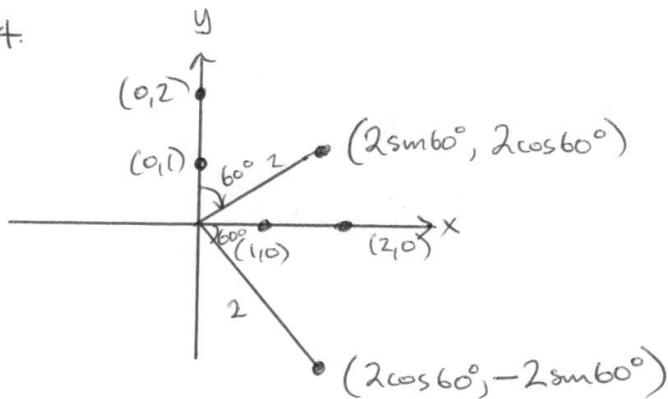
$$= \textcircled{2} \times \textcircled{4} - \textcircled{3} \times \textcircled{4}$$

$$= \frac{1}{3} x^3 \ln x - \int \frac{1}{x} \cdot \frac{x^3}{3} \, dx$$

Dif	INT
$\textcircled{2}$ $\ln x$	x^2
$\textcircled{3}$ $\frac{1}{x}$	$\frac{x^3}{3}$

$$\begin{aligned} I &= \frac{x^3}{3} \ln x - \int \frac{x^2}{3} dx \\ &= \frac{x^3}{3} \ln x - \frac{1}{9} x^3 + C \end{aligned}$$

4.



$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$



$$\begin{pmatrix} 2 \cos 60^\circ & 2 \sin 60^\circ \\ -2 \sin 60^\circ & 2 \cos 60^\circ \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

5.

$$\begin{aligned} \binom{n+1}{3} - \binom{n}{3} &= \frac{(n+1)!}{3!(n+1-3)!} - \frac{n!}{3!(n-3)!} \\ &= \frac{(n+1)!}{(n-2)! \times 3!} - \frac{n!}{(n-3)! \times 3!} \\ &= \frac{(n+1)(n)(n-1)}{6} - \frac{n(n-1)(n-2)}{6} \\ &= \frac{1}{6} n(n-1) [(n+1) - (n-2)] = \frac{1}{6} n(n-1)(3) \\ \text{LHS} &= \frac{1}{2} n(n-1) \end{aligned}$$

$$\begin{aligned}
 RHS &= \binom{n}{2} = \frac{n!}{(n-2)! 2!} = \frac{n(n-1)}{2 \times 1} \\
 &= \frac{1}{2} n(n-1)
 \end{aligned}$$

$$LHS = RHS$$

6. $\underline{u} \cdot (\underline{v} \times \underline{w})$ Scalar triple product

$$\begin{array}{ccccc}
 -2 & 3 & -1 & -2 & 3 \\
 0 & \cancel{2} & \cancel{1} & \cancel{0} & 2 \\
 5 & -1 & 4 & \cancel{5} & -1
 \end{array}$$

$$\begin{aligned}
 & (-16 + 15 + 0) - (-10 + 2 + 0) \\
 & (-1) - (-8) = 7
 \end{aligned}$$

$$7 \quad \text{Let } \frac{3x+5}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$3x+5 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$$x = -2 \rightarrow -1 = B(-1)(1) \quad (B = 1)$$

$$x = -3 \rightarrow -4 = C(-2)(-1) \quad (C = -2)$$

$$x = -1 \rightarrow 2 = A(1)(2) \quad (A = 1)$$

$$\begin{aligned}
 I &= \int_1^2 \frac{1}{x+1} dx + \int_1^2 \frac{1}{x+2} dx - \int_1^2 \frac{2}{x+3} dx \\
 &= [\ln(x+1)]_1^2 + [\ln(x+2)]_1^2 - [2\ln(x+3)]_1^2 \\
 &= \cancel{\ln 3 - \ln 2} + \cancel{\ln 4 - \ln 3} - 2\ln 5 + 2\ln 4 \\
 &= 3\ln 4 - \ln 2 - 2\ln 5 \\
 &= \ln 4^3 - \ln 2 - \ln 5^2 = \ln \left(\frac{64}{2 \times 25} \right) \\
 &= \ln \left(\frac{32}{25} \right)
 \end{aligned}$$

8 (a) typical odd integers $\frac{m=2s+1}{n=2r+1}$ $s, r \in \mathbb{N}$

$$\begin{aligned}
 mn &= (2s+1)(2r+1) = 4sr + 2s + 2r + 1 \\
 &= 2(2sr + s + r) + 1 \\
 &= 2(\text{something}) + 1
 \end{aligned}$$

$\Rightarrow mn$ is odd also

(b) p^n is odd.

$$n=1 \quad p^1 = p \quad \text{which is odd (given)}$$

Assume p^k is odd.

$\Rightarrow p \cdot p^k = p^{k+1}$ and this is the product of 2 odd integers $\Rightarrow p^{k+1}$ is odd (using (a))

hence p^n is odd when $n=1$

p^{k+1} is odd whenever p^k is odd.

$\rightarrow p^n$ is odd for all positive integers

$n \geq 1$ by induction

$$9. f(x) = 1 + (\sin x)^2 \quad f(0) = 1$$

$$\begin{aligned} f'(x) &= 2 \sin x \cos x \\ &= \sin 2x \end{aligned} \quad f'(0) = 0$$

$$f''(x) = 2 \cos 2x \quad f''(0) = 2$$

$$f'''(x) = -4 \sin 2x \quad f'''(0) = 0$$

$$f^{(4)}(x) = -8 \cos 2x \quad f^{(4)}(0) = -8$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \dots$$

$$f(x) = 1 + \frac{2x^2}{2!} - \frac{8x^4}{4!}$$

$$f(x) = 1 + x^2 - \frac{1}{3}x^4$$

10. Curve not symmetrical about y-axis

$\Rightarrow f(x)$ is not even.

Curve not symmetrical about origin

$\Rightarrow f(x)$ is not odd.

$\Rightarrow f(x)$ is neither even nor odd.

11.

$$x^2 + 4x + 5 = 0$$

Does not factorise so

$$b^2 - 4ac = 16 - 20 = -4$$

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$$

$$\lambda = -2 \pm i$$

gen solⁿ $y = e^{-2x} (A \cos x + B \sin x)$

$$y=3 \quad x=0$$

$$3 = A \cos 0$$

$$(A = 3)$$

$$y = e^{-\pi} \quad x = \frac{\pi}{2}$$

$$e^{-\pi} = e^{-\pi} (A \cos \frac{\pi}{2} + B \sin \frac{\pi}{2})$$

$$e^{-\pi} = e^{-\pi} \cdot B$$

$$(B = 1)$$

Particular
Sln

$$y = e^{-2x} (3 \cos x + \sin x)$$

12. Suppose

x is rational $\Rightarrow 2+x$ is irrational

$$\Rightarrow x = \frac{m}{n} \quad m, n \text{ integers}$$

$$\begin{aligned}\Rightarrow 2+x &= 2 + \frac{m}{n} \\ &= \frac{2n+m}{n}\end{aligned}$$

$\Rightarrow 2+x$ is rational

Contradiction!

Original supposition false

hence x is irrational $\Rightarrow 2+x$ is irrational

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$$y = t^3 - \frac{5}{2}t^2 \quad x = t^{3/2}$$

$$\dot{y} = 3t^2 - 5t \quad \dot{x} = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}} = \frac{1}{2t^{1/2}}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{\dot{y}}{\dot{x}} = \frac{3t^2 - 5t}{\frac{1}{2\sqrt{t}}} = 2\sqrt{t}(3t^2 - 5t) \\ &= 6t^{5/2} - 10t^{3/2} = 2t^{3/2}(3t - 5)\end{aligned}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{15t^{3/2} - 15t^{1/2}}{\frac{1}{2}t^{-1/2}}$$

$$\begin{aligned}
 &= 2t^{\frac{1}{2}}(15t^{\frac{3}{2}} - 15t^{\frac{1}{2}}) \\
 &= 30t^2 - 30t \\
 &= at^2 + bt \quad \begin{matrix} a = 30 \\ b = -30 \end{matrix}
 \end{aligned}$$

possible PI when $\frac{d^2y}{dx^2} = 0$

$$\Rightarrow 30t^2 - 30t = 0$$

$$30t(t-1) = 0$$

$$\begin{matrix} t=0 \\ \text{invalid} \end{matrix} \quad \text{or} \quad t=1 \\
 \text{as } t>0$$

point of inflection at $t=1$

$$\text{at point } x = \sqrt{1} = 1$$

$$y = 1^3 - \frac{5}{2} \times 1^2 = 1 - \frac{5}{2} = -\frac{3}{2}$$

$$\begin{matrix} \text{point } (1, -\frac{3}{2}) \\ a, b \end{matrix} \quad \begin{matrix} \text{gradient} \\ \text{of tangent} \end{matrix} \rightarrow \begin{matrix} \frac{dy}{dx} = 6 - 10 \\ \text{at } t=1 \end{matrix} = -4 \text{ m.}$$

$$y - b = m(x - a)$$

$$y + \frac{3}{2} = -4(x - 1)$$

$$y = -4x + \frac{5}{2}$$

$$14. \left(\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & -1 & a & 2 & 1 \end{array} \right) \quad R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - 2R_1$$

$$\left(\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 1 & a-2 & 0 & 0 \end{array} \right) \quad R_3 \rightarrow 2R_3 - R_2$$

$$\left(\begin{array}{ccc|cc} 1 & -1 & 1 & 1 & 1 \\ 0 & 2 & 1 & -1 & 0 \\ 0 & 0 & 2a-5 & 1 & 1 \end{array} \right)$$

unique solution provided $2a-5 \neq 0$
 $\Rightarrow 2a \neq 5$
 $\Rightarrow a \neq 2.5$

If $a = 2.5$ bottom row becomes

$$0.x + 0.y + 0.z = 1$$

which is an impossibility, no solution possible

When $a=3$

$$(z=1)$$

$$2y + z = -1 \Rightarrow 2y = -2 \Rightarrow y = -1$$

$$x - y + z = 1 \quad x + 1 + 1 = 1$$

$$(x=-1)$$

14 cont

Solution for $a = 3 \rightarrow (-1, -1, 1)$

$$AB = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$C\tilde{\underline{x}} = B$ also gives solution $\tilde{\underline{x}} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

So $\tilde{\underline{x}} = AB$ and $C\tilde{\underline{x}} = B$

$$\Rightarrow AC\tilde{\underline{x}} = AB$$

$$\tilde{\underline{x}} = AB \quad AC\tilde{\underline{x}} = AB$$

$$\Rightarrow AC = I.$$

$\Rightarrow A$ and C are inverses

15.

$$\begin{array}{ll} y = x^2 & \rightarrow \\ y^2 = 8x & \rightarrow \end{array} \begin{array}{l} y^2 = x^4 \\ y^2 = 8x \end{array}$$

Meet $y^2 = y^2 \rightarrow x^4 = 8x$
 $x^4 - 8x = 0$
 $x(x^3 - 8) = 0$

$$\begin{array}{c} x=0, \quad x=2 \\ \swarrow \qquad \searrow \\ (0,0) \qquad (2,4) \end{array}$$

Area between 2 curves

$$\begin{aligned} & \int_0^2 \left(\underset{\text{upper}}{y} - \underset{\text{lower}}{y} \right) dx \\ &= \int_0^2 (\sqrt{8}x^{1/2} - x^2) dx \\ &= \left[\frac{2}{3}\sqrt{8}x^{3/2} - \frac{1}{3}x^3 \right]_0^2 \\ &= \left(\frac{2}{3}\sqrt{8} \times 2\sqrt{2} - \frac{8}{3} \right) - 0 \\ &= \frac{2}{3} \times 8 - \frac{8}{3} = \frac{8}{3} = 2\frac{2}{3} \text{ units}^2 \end{aligned}$$

Complete design

$$\begin{aligned} 2\frac{2}{3} \times 4 &= \frac{32}{3} \text{ units}^2 \\ &= 10\frac{2}{3} \text{ units}^2 \end{aligned}$$

15 cont

$$\text{Volume} = \pi \int x_1^2 dy - \pi \int x_2^2 dy$$

where $x_1^2 = y$ $8x_2 = y^2$
 $x_2 = \frac{y^2}{8}$
 $x_2^2 = \frac{y^4}{64}$

$$\text{Volume} = \pi \int_0^4 y^2 dy - \frac{\pi}{64} \int_0^4 y^4 dy$$

$$= \pi \left[\frac{y^3}{3} \right]_0^4 - \pi \left[\frac{y^5}{320} \right]_0^4$$

$$= 8\pi - \frac{1024}{320}\pi = 4.8\pi \approx 15.1 \text{ units}^3$$

$$16 \quad z^3 = r^3 (\cos 3\theta + i \sin 3\theta)$$

$$\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3 = \cos 2\pi + i \sin 2\pi = 1$$

So

$$8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)^3 = 8$$

$$\begin{aligned} \Rightarrow 1 \text{ solution is } z &= 2 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \\ &= 2 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= \underline{-1 + \sqrt{3}i} \end{aligned}$$

conjugate
also solution

$$z = -1 - \sqrt{3}i$$

$$\text{other solution } z = 2$$

$$\text{So } \underline{z_1 = 2, z_2 = -1 + \sqrt{3}i, z_3 = -1 - \sqrt{3}i}$$

$$\begin{aligned} (a) \quad z_1 + z_2 + z_3 &= 2 + (-1 + \sqrt{3}i) + (-1 - \sqrt{3}i) \\ &= \underline{0} \end{aligned}$$

$$\begin{aligned} (b) \quad z_1^6 + z_2^6 + z_3^6 &= 2^6 + (-1 + \sqrt{3}i)^6 + (-1 - \sqrt{3}i)^6 \\ &= 2^6 + 64 + 64 \\ &= \underline{64 + 64 + 64} \end{aligned}$$

$$\underline{= 192}$$