Content List

Mathematics 1 (AH)

1.1 Algebra.

1.1.1 know and use the notation $n!$, ${}^nC_r$ and $\binom{n}{r}$

1.1.2 know the results $\binom{n}{r} = \binom{n}{n-r}$ and

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

1.1.3 know Pascal's triangle. Pascal's triangle should be extended up to $n = 7$.

1.1.4 know and use the binomial theorem

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r \text{, for } r,\ n \in N$$

e.g. expand $(2u - 3v)^5$ **[A/B]**

1.1.5 evaluate specific terms in a binomial expansion e.g.

x^5 in $(x+3)^7$ **e.g.** x^7 in $\left(x + \frac{2}{x}\right)^9$ **[A/B]**

1.1.6 express a proper rational function as a sum of partial fractions (denominator of degree at most 3 and easily factorised).

include cases where an improper rational function is reduced to a polynomial and a proper rational function by division or otherwise. [A/B]

1.2 Differentiation.

1.2.1 know the meaning of the terms limit, derivative, differentiable at a point, differentiable on an interval, derived function, second derivative.

1.2.2 use the notation: $f'(x), f''(x), \frac{dy}{dx}, \frac{d^2y}{dx^2}$

1.2.3 recall the derivatives of x^{α} (α rational), $\sin x$ and $\cos x$

1.2.4 know and use the rules for differentiating linear sums, products, quotients and composition of functions:

$$(f(x) + g(x))' = f'(x) + g'(x);$$

$$(kf(x))' = kf'(x), \text{ where } k \text{ is a constant;}$$

the chain rule: $(f(g(x)))' = f'(g(x))\,g'(x)$;

the product rule:

$$(f(x)\,g(x))' = f'(x)\,g(x) + f(x)\,g'(x);$$

the quotient rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

Differentiate given functions which require more than one application of one or more of

the chain rule, product rule and the quotient rule. [A/B]

1.2.5 know

- the derivative of $\tan x$
- the definitions and derivatives of $\sec x$, $\operatorname{cosec} x$ and $\cot x$
- the derivatives of e^x $(\exp x)$ and $\ln x$

1.2.6 know the definition $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

1.2.7 know the definition of higher derivatives $f^n(x), \frac{d^n y}{dx^n}$

1.2.8 apply differentiation to:

a) rectilinear motion

b) extrema of functions: the maximum and minimum values of a continuous function *f* defined on a closed interval [*a, b*] can occur at stationary points, **end points or points where *f′* is not defined. [A/B]**

c) optimisation problems

1.3 Integration

1.3.1 know the meaning of the terms integrate, integrable, integral, indefinite integral, definite integral and constant of integration.

1.3.2 recall standard integrals of

x^{α} $(\alpha \in Q, \alpha \neq -1)$, $\sin x$ and $\cos x$ and know the following

$$\int (af(x) + bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx, \quad a, b \in R$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \ a < c < b$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx, \ b \neq a$$

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x)$$

1.3.3 know the integrals of $e^x, x^{-1}, \sec^2 x$.

1.3.4 integrate by substitution: expressions requiring a simple substitution

candidates are expected to integrate simple functions on sight

expressions where the substitution will be given e.g.

$$\int \cos^3 x \sin x\,dx, \ u = \cos x$$

e.g. $\int \frac{6 \sin x}{\sqrt{1 - 4\cos^2 x}}\,dx, \ u = \cos x$ **[A/B]**

the following special cases of substitution

$$\int f(ax+b)\,dx, \ \int \frac{f'(x)}{f(x)}\,dx$$

1.3.5 use an elementary treatment of the integral as a

limit using rectangles

1.3.6 apply integration to the evaluation of areas **including integration with respect to y. Other applications may include (i) volumes of simple solids of revolution (disc/washer method)**

(ii) speed/time graph. [A/B]

1.4 Properties of Functions

1.4.1 know the meaning of the terms function, domain, range, inverse function, critical point, stationary point, point of inflexion, concavity, local maxima and minima, global maxima and minima, continuous, discontinuous, asymptote.

1.4.2 determine the domain and the range of a function.

1.4.3 use the derivative tests for locating and identifying stationary points i.e. concave up $\Leftrightarrow f''(x) > 0$, concave down $\Leftrightarrow f''(x) < 0$, a necessary and sufficient condition for a point of inflexion is a change in concavity.

1.4.4 sketch the graphs of $\sin x, \cos x, \tan x, e^x, \ln x$ and their inverse functions, simple polynomial functions.

1.4.5 know and use the relationship between the graph of $y = f(x)$ and the graphs of $y = kf(x)$, $y = f(x) + k$, $y = f(x+k)$, $y = f(kx)$,

where k is a constant.

1.4.6 know and use the relationship between the graph of $y = f(x)$ and the graphs of $y = |f(x)|$, $y = f^{-1}(x)$

1.4.7 given the graph of a function f, sketch the graph of a related function.

1.4.8 determine whether a function is symmetrical, even or odd or neither and use these properties in graph sketching.

1.4.9 sketch graphs of real rational functions using available information, derived from calculus and/or algebraic arguments, on zeros, asymptotes (vertical and non-vertical), critical points, symmetry.

1.5 Systems of Linear Equations

1.5.1 use of the introduction of matrix ideas to organise a system of linear equations

1.5.2 know the meaning of the terms matrix, element, row, column, order of a matrix, augmented matrix.

1.5.3 use elementary row operations (EROs)

Reduce to upper triangular form using EROs.

1.5.4 solve a 3×3 system of linear equations using Gaussian elimination on an augmented matrix

1.5.5 find the solution of a system of linear equations $Ax = b$, where A is a square matrix, include cases of unique solution, **no solution (inconsistency) and an infinite family of solutions [A/B]**

1.5.6 know the meaning of the term ill-conditioned [A/B]

1.5.7 compare the solutions of related systems of two equations in two unknowns and recognise ill-conditioning [A/B]

Mathematics 2 (AH)

2.1 Further Differentiation

2.1.1 know the derivatives of $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$

2.1.2 differentiate any inverse function using the technique:

$$y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow \left(f^{-1}(x)\right)' f'(y) = 1, \; etc.,$$

and know the corresponding result $\dfrac{dy}{dx} = \dfrac{1}{dx/dy}$ **[A/B]**

2.1.3 understand how an equation $f(x, y) = 0$ defines y implicitly as one (or more) function(s) of x.

2.1.4 use implicit differentiation to find first **and second derivatives [A/B]**

2.1.5 use logarithmic differentiation, recognising when it is appropriate in extended products and quotients and indices involving the variable. [A/B]

2.1.6 understand how a function can be defined parametrically.

2.1.7 understand simple applications of parametrically defined functions e.g. $x^2 + y^2 = r^2, \; x = r\cos\theta, \; y = r\sin\theta$

2.1.8 use parametric differentiation to find first **and second derivatives [A/B],** and apply to motion in a plane

2.1.9 apply differentiation to related rates in problems where the functional relationship is given explicitly or implicitly

2.1.10 solve practical related rates by first establishing a functional relationship between appropriate variables [A/B]

2.2 Further Integration

2.2.1 know the integrals of $\dfrac{1}{\sqrt{1-x^2}}, \dfrac{1}{1+x^2}$; use the substitution $x = at$ to integrate functions of the form

$$\frac{1}{\sqrt{a^2 - x^2}}, \frac{1}{a^2 + x^2}$$

integrate rational functions, both proper and **improper [A/B]**, by means of partial fractions; the degree of the denominator being ≤ 3; the denominator may include:

(i) two separate or repeated linear factors

(ii) three linear factors with constant numerator **and with non-constant numerator [A/B]**

(iii) **a linear factor and an irreducible quadratic factor of the form $x^2 + a$ [A/B]**

2.2.2 integrate by parts with one application.

2.2.3 integrate by parts involving repeated applications [A/B]

2.2.4 know the definition of a differential equation and the meaning of the terms linear, order, general solution, arbitrary constants, particular solution, initial condition

2.2.5 solve first order differential equations (variables separable)

2.2.6 formulate a simple statement involving rate of change as a simple separable first order differential equation, including the finding of a curve in the plane, given the equation of the tangent at *(x, y),* which passes through a given point

2.2.7 know the laws of growth and decay: applications in practical contexts

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2.3 Complex Numbers

2.3.1 know the definition of i as a solution of $x^2 + 1 = 0$, so that $i = \sqrt{-1}$

2.3.2 know the definition of the set of complex numbers as $C = \{a + ib : a, b \in R\}$

2.3.3 know the definition of real and imaginary parts

2.3.4 know the terms complex plane, Argand diagram

2.3.5 plot complex numbers as points in the complex plane

2.3.6 perform algebraic operations on complex numbers:

equality (equating real and imaginary parts), addition, subtraction, multiplication and division

2.3.7 evaluate the modulus, argument and conjugate of complex numbers

2.3.8 convert between Cartesian and polar form.

2.3.9 know the fundamental theorem of algebra and the conjugate roots property

2.3.10 factorise polynomials with real coefficients

2.3.11 solve simple equations involving a complex variable by equating real and imaginary parts

e.g. solve $z + i = 2\overline{z} + 1$, **solve $z^2 = 2\overline{z}$ [A/B]**

2.3.12 interpret geometrically certain equations or inequalities in the complex plane e.g. $|z| = 1$; $|z - a| = b$; $|z - 1| = |z - i|$; **$|z - a| > b$ [A/B]**

2.3.13 know and use de Moivre's theorem with positive integer indices **and fractional indices [A/B]**

2.3.14 apply de Moivre's theorem to multiple angle trigonometric formulae [A/B].

2.3.15 apply de Moivre's theorem to find *n*th roots of unity [A/B].

2.4 Sequences and Series

2.4.1 know the meaning of the terms infinite sequence, infinite series, *n*th term, sum to *n* terms (partial sum), limit, sum to infinity (limit to infinity of the sequence of partial sums), common difference, arithmetic sequence, common ratio, geometric sequence, recurrence relation

2.4.2 know and use the formulae $u_n = a + (n-1)d$ and $S_n = \frac{1}{2}n[2a + (n-1)d]$ for the *n*th term and the sum to *n* terms of an arithmetic series, respectively

2.4.3 know and use the formulae $u_n = ar^{n-1}$ and $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$, for the *n*th term and the sum to *n* terms of a geometric series, respectively

2.4.4 know and use the condition on *r* for the sum to infinity to exist and the formula $S_\infty = \frac{a}{1-r}$ for the sum to infinity of a geometric series where $|\boldsymbol{r}| < \mathbf{1}$ **[A/B]**

2.4.5 expand $\frac{1}{1-r}$ as a geometric series

and tend to $\boldsymbol{\frac{1}{a+b}}$ **[A/B]**

2.4.6 know the sequence $\left(1+\frac{1}{n}\right)^n$ and its limit.

2.4.7 know and use the $\sum$ notation.

2.4.8 know the formula $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and apply it to simple sums e.g. $\sum_{r=1}^{n}(ar+b) = a\sum_{r=1}^{n} r + \sum_{r=1}^{n} b$

2.5 Elementary Number Theory and Methods of Proof

2.5.1 understand the nature of mathematical proof

2.5.2 understand and make use of the notations =>, <= and <=> know the corresponding terminology implies, implied by, equivalence

2.5.3 know the terms natural number, prime number, rational number, irrational number

2.5.4 know and use the fundamental theorem of arithmetic

2.5.5 disprove a conjecture by providing a counter-example

2.5.6 use proof by contradiction in simple examples

2.5.7 use proof by mathematical induction in simple examples

2.5.8 prove the following results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$;

the binomial theorem for positive integers;

de Moivre's theorem for positive integers

3.1 Vectors

3.1.1 know the meaning of the terms position vector, unit vector, scalar triple product, vector product, components, direction ratios/cosines

3.1.2 calculate scalar and vector products in three dimensions

3.1.3 know that $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

3.1.4 find $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} . \mathbf{b} \times \mathbf{c}$ in component form

3.1.5 know the equation of a line in vector form, parametric and symmetric form

3.1.6 know the equation of a plane in vector form, parametric and symmetric form, Cartesian form

3.1.7 find the equations of lines and planes given suitable defining information

3.1.8 find the angles between two lines, two planes and between a line and a plane [A/B]

3.1.9 find the intersection of two lines, a line and a plane and two or three planes [A/B]

3.2 Matrix Algebra

3.2.1 know the meaning of the terms: matrix, element, row, column, order, identity matrix, inverse, determinant, singular, non-singular, transpose

3.2.2 perform matrix operations: addition, subtraction, multiplication by a scalar, multiplication, establish equality of matrices

3.2.3 know the properties of the operations: $A + B = B + A$; $AB \neq BA$ in general; $(AB)C = A(BC)$;

$A(B + C) = AB + AC$; $(A')' = A$;

$(A + B)' = A' + B'$; $(AB)' = B'A'$;

$(AB)^{-1} = B^{-1}A^{-1}$;

$\det(AB) = \det A \det B$

3.2.4 calculate the determinant of 2×2 and 3×3 matrices

3.2.5 know the relationship of the determinant to invertability

3.2.6 find the inverse of a 2×2 matrix

3.2.7 find the inverse, where it exists, of a 3×3 matrix by elementary row operations

3.2.8 know the role of the inverse matrix in solving linear systems

3.2.9 use 2×2 matrices to represent geometrical transformations in the (x, y) plane

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3.3 Further Sequences and Series

3.3.1 know the term power series

3.3.2 understand and use the Maclaurin series:

$$f(x) = \sum_{r=0}^{\infty} \frac{x^r}{r!} f^{(r)}(0)$$

3.3.3 find the Maclaurin series of simple functions: e^x, $\sin x$, $\cos x$, $\tan^{-1} x$, $(1 + x)^{\alpha}$, $\ln(1 + x)$, knowing their range of validity

3.3.4 find the Maclaurin expansions for simple composites, e.g. e^{2x}, **e.g. $e^{\sin x}$, $e^x \cos 3x$ [A/B]**

3.3.5 use the Maclaurin series expansion to find power series for simple functions to a stated number of terms

3.3.6 use iterative schemes of the form $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$ to solve equations where $x = g(x)$ is a rearrangement of the original equation

3.3.7 use graphical techniques to locate an approximate solution x_0

3.3.8 know the condition for convergence of the sequence $\{x_n\}$ given by $x_{n+1} = g(x_n)$, $n = 0, 1, 2, \ldots$

3.4 Further Ordinary Differential Equations

3.4.1 solve first order linear differential equations using the integrating factor method

3.4.2 find general solutions and solve initial value problems

3.4.3 know the meaning of the terms: second order linear differential equation with constant coefficients, homogeneous, non-homogeneous, auxiliary equation, complementary function and particular integral

3.4.4 solve second order homogeneous ordinary differential equations with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

3.4.5 find the general solution in the three cases where the roots of the auxiliary equation:

(i) are real and distinct.

(ii) coincide (are equal) [A/B]

(iii) are complex conjugates [A/B]

3.4.6 solve initial value problems

3.4.7 solve second order non - homogeneous ordinary differential equations with constant coefficients

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ using the auxiliary equation and particular integral method [A/B]

3.5 Further Number Theory and Further Methods of Proof

3.5.1 know the terms necessary condition, sufficient condition, if and only if, converse, negation

3.5.2 use further methods of mathematical proof: some simple examples involving the natural numbers

3.5.3 direct methods of proof: sums of certain series and other straightforward results

3.5.4 further proof by contradiction

3.5.5 further proof by mathematical induction

prove the following result

$$\sum_{r=1}^{n} r^2 = \frac{1}{6} n(n+1)(2n+1);\ n \in N$$

3.5.6 know the result $\sum_{r=1}^{n} r^3 = \frac{1}{4} n^2 (n+1)^2$

3.5.7 apply the above results and the one for $\sum_{r=1}^{n} r$ to prove by direct methods results concerning other sums

e.g. $\sum_{r=1}^{n} r(r+1) = \frac{1}{3} n(n+1)(n+2)$.

e.g. $\sum_{i=1}^{4} (3i+1)$.

e.g.

$$\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4} n(n+1)(n+2)(n+3)$$

e.g. $\boldsymbol{\sum_{r=1}^{n} r(r^2+2)}$ **[A/B]**

3.5.8 know the division algorithm and proof

3.5.9 use Euclid's algorithm to find the greatest common divisor (g.c.d.) of two positive integers.

3.5.10 know how to express the g.c.d. as a linear combination of the two integers [A/B]

3.5.11 use the division algorithm to write integers in terms of bases other than 10 [A/B]