TeeJay Publisher's

Advanced Higher

PUPIL

Revision Notes

A resume of what you should know for the exam

Mathematics 1

Mathematics 1

Unit 1 *Binomial Theorem / Partial Fractions*

A. Know how to find the numbers in Pascals Triangle.

B. Know how to expand $(x+y)^n$ using Pascal's Triangle.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$
(etc.)

C. Know the meaning of factorial n (n factorial) = $n! = n(n-1)(n-2) \ldots 3.\ 2.\ 1$

D. Know the Binomial coefficient $\binom{n}{r}$ $(= {}^nC_r) = \dfrac{n!}{r!(n-r)!}$ e.g. $\binom{10}{7} = \dfrac{10\times 9\times 8}{3\times 2\times 1}$.

E. Know the Binomial Theorem => $(x+y)^n = \sum_{r=0}^{r=n}\binom{n}{r}x^{n-r}y^r$.

F. Know also $\binom{n}{r}+\binom{n}{r+1}=\binom{n+1}{r}$ and $\sum_{r=0}^{r=n}\binom{n}{r}=2^n$.

G. Understand how to expand $(x+y+z)^n = (x+(y+z))^n = \sum_{r=o}^{r=n}\binom{n}{r}x^{n-r}(y+z)^r$.

H. Know the <u>General Term</u> in $(x+y)^n$ is $T_{r+1}=\binom{n}{r}x^{n-r}y^r$ (useful to find single terms).

I. Partial Fractions (a) $\dfrac{x+1}{(x+3)(x-2)} = \dfrac{A}{(x+3)} + \dfrac{B}{(x-2)}$ etc.

(b) $\dfrac{x+1}{(x+3)(x^2-2x+3)} = \dfrac{A}{(x+3)} + \dfrac{Bx+C}{(x^2-2x+3)}$ etc.

(c) $\dfrac{x+1}{(x+3)(x-2)^2} = \dfrac{A}{(x+3)} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$ etc.

*Note - Remember to divide if the degree of numerator is greater than or equal to the degree of the denominator.

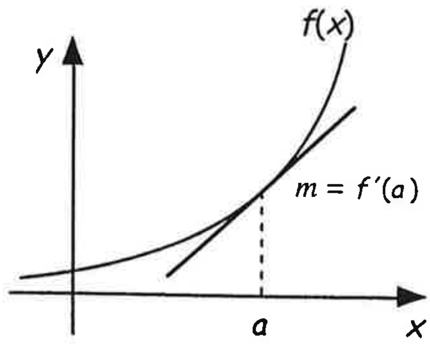
Mathematics 1

Unit 2 *Differentiation 1*

A. Know how to differentiate simple functions from First Principles.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Know that $f'(a)$ is simply the gradient of the tangent to curve $f(x)$ at $x = a$.



B. New Trig Functions :-

(i) $\sec x = \dfrac{1}{\cos x}$ (ii) $\operatorname{cosec} x = \dfrac{1}{\sin x}$ (iii) $\cot x = \dfrac{1}{\tan x}$.

C. Standard Derivatives

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan(ax+b)$	$a\sec^2(ax+b)$
$\sec x$	$\sec x\tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \operatorname{cotan} x$
$\cot x$	$-\operatorname{cosec}^2 x$
$e^{(ax+b)}$	$ae^{(ax+b)}$
$\ln(ax+b)$	$\dfrac{a}{(ax+b)}$

D. Three Main Rules for differentiation:-

Chain Rule $\dfrac{d}{dx}(g(f(x))) = g'(f(x))\times f'(x)$

Product Rule $\dfrac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + g(x)f'(x)$

Quotient Rule $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

E. Know how to find 2nd and Higher derivatives

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad \frac{d^ny}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}(\ldots\ldots)\right)\right)\right)$$

or $f''(x)$, $f'''(x)$, $f^n(x)$ etc.

F. Know how to apply Second Derivative test to decide on nature of turning points.
if $x = a$ is a stationary value, then:-

if $f''(a) > 0$ => (a minimum T.P.) and if $f''(a) < 0$ => (a maximum T.P.)

G. Displacement, Velocity, Acceleration and derivatives.
If $x(t)$ if a (distance) function of time, then
$x'(t)$ gives the speed (velocity) and
$x''(t)$ gives the acceleration of the object.

H. Optimisation :-

If you are asked to find the largest, smallest, greatest, least, best, cheapest etc.
=> find the derivative, set it equal to zero and solve for the variable.

This should enable you to optimise the solution.

Mathematics 1

Unit 3 *Integration 1*

A. Standard Integrals

$f(x)$	$\int f(x)dx$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b) + c$
$\text{cosec}^2(ax+b)$	$-\frac{1}{a}\cot(ax+b) + c$
$\tan x$	$\ln(\lvert\cos x\rvert) + c$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln(ax+b) + c$

B. Integration by Substitution:-

Type 1 $\int \cos x \sin^3 x dx$ let $u = \sin x$ => $du = \cos x\, dx$ etc

Type 2 $\int \frac{x}{\sqrt{9-x^2}}dx$ let $x = 3\sin\theta$ => $dx = 3\cos\theta\, d\theta$ etc.

Special Type $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$

C. Area under curve $\int_a^b f(x)dx$ = area under $f(x)$ from a to b.

Area between curves $\int_a^b (f(x)-g(x))dx$ = area between $f(x)$ and $g(x)$ from a to b.

Area between curve and y-axis $\int_{y=a}^{y=b} f(y)dy = \int_{y=a}^{y=b} x dy$ = area between $f(x)$ and y-axis.

Volume of Solid of Revolution about x-axis $V = \pi\int_a^b \left(f(x)\right)^2 dx$.

Mathematics 1

Unit 4 *Functions and Related Graphs*

A. A Rational Function is one of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ & $Q(x)$ are polynomials.

B. Asymptotes

(a) vertical — if $Q(x) = 0$ has a solution $x = a$, then $x = a$ is a V.A.

Check what happens as $x \rightarrow a^+$ and $x \rightarrow a^-$

(b) horizontal — if degree of $P(x)$ < degree of $Q(x)$, then the y-axis is a horizontal asymptote. (H.A.)

Check what happens as $x \rightarrow +\infty$ and $x \rightarrow -\infty$.

if degree of $P(x)$ = degree of $Q(x)$, then there is a horizontal asymptote ($y = k$)

Divide each term by the highest power of x and check what happens as $x \rightarrow +\infty$ and $x \rightarrow -\infty$.

(c) oblique — if degree of $P(x)$ is <u>exactly 1 more than</u> the degree $Q(x)$, then there is an oblique (sloping) asymptote and it is found by dividing $P(x)$ by $Q(x)$.

Then check what happens as $x \rightarrow +\infty$ and $x \rightarrow -\infty$.

C. To sketch $f(x) = \frac{x^3 - 1}{(x+1)(x-2)}$

find where it cuts y-axis (set $x = 0$) => $(0, \frac{1}{2})$

find where it cuts x-axis (set $y = 0$ & solve) => $(1,0)$

find the stationary points (set $f'(x) = 0$ and solve) => (?,...) and (?,...)

find the vertical asymptotes ($x = -1$ and $x = 2$) (check $x \rightarrow -1^{\pm}$ and $2^{\pm}$)

find any horizontal asymptote (as deg ($P(x)$ > deg $Q(x)$ —> no H.A.)

find any sloping asymptote $y = x + 1$ and study what happens as $x \rightarrow \pm\infty$

then sketch $f(x)$ using all the above information

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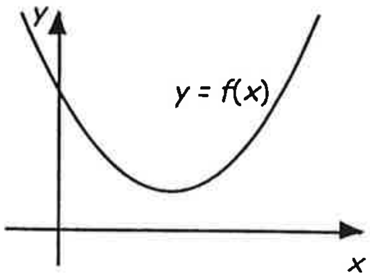
D. Be able to sketch graphs "related" to $f(x)$ if you know what $f(x)$ looks like.



(i) $g(x) = kf(x)$ multiplies height of $f(x)$ by k at each point. ↕

(ii) $g(x) = f(kx)$ reduces distance from y-axis to graph by $^1/_k$ for each point on $f(x)$. ↔

(iii) $g(x) = f(x + k)$ moves graph horizontally by k units (left if $k > 0$).

(iv) $g(x) = f(x) + k$ moves graph up (if $k > 0$) or down (if $k < 0$).

(v) $g(x) = -f(x)$ reflects $f(x)$ over x-axis.

(vi) $g(x) = f^{-1}(x)$ reflects $f(x)$ over the line $y = x$.

(vii) $g(x) = |f(x)|$ reflects the bits of $f(x)$ lying below the x-axis over the x-axis.

(viii) $g(x) = f'(x)$ see 5th year method. Stationary points of $f(x)$ become zeros of $f'(x)$and slope of $f(x)$ determines positive or negative nature of $f'(x)$.

E. Even and Odd Functions

(a) A function $f(x)$ is EVEN if $f(a) = f(-a)$ for every value a where $f(x)$ is defined. (symmetric when reflected over the y axis)

(b) A function $f(x)$ is ODD if $f(a) = -f(-a)$ for every value a where $f(x)$ is defined. (symmetric when rotated 180° around the origin)

Mathematics 1

Unit 5 *Matrices 1*

A. Know the terms "order" (e.g. 3 by 4) of a matrix and "element" of a matrix.

B. Know what the transpose of matrix A is A^T or A'. if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \Rightarrow A' = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

C. Solve 3 x 3 systems of equations by Gaussian elimination.

D. Know what is meant by a system of equations being ill-conditioned.
i.e. a small change in one of the coefficients => a large change in the solutions.

Mathematics 2

Mathematics 2

Unit 1 *Further Differentiation*

A. More Standard Derivatives

$f(x)$	$f'(x)$
$\sin^{-1}x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}x$	$\frac{1}{1+x^2}$

B. Be able to use Implicit Differentiation.

derivative of $x^2 - 3xy + y^2 = 1$ is

$$2x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} = 0 \quad *$$

$$\Rightarrow \quad (2y-3x)\frac{dy}{dx} = 3y - 2x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{3y-2x}{(2y-3x)} \quad **$$

C. Be able to find second derivative, $\frac{d^2y}{dx^2}$ from * or ** above.

D. By change of variable, be able to solve differential equation problems.

e.g. Volume of a balloon is increasing by 160 cm^3 per second. $(\frac{dV}{dt} = 160)$

Find rate of change of radius when $r = 5$ cm.

=> Find $\frac{dr}{dt} = \left(\frac{dr}{dV} \times \frac{dV}{dt}\right)$, but since $V = \frac{4}{3}\pi r^3$ => $\frac{dV}{dr} = 4\pi r^2$

=> Hence, $\frac{dr}{dt} = \frac{1}{4\pi r^2} \times 160 = \frac{40}{\pi r^2} = \frac{40}{\pi \times 5 \times 5} = 0{\cdot}51$ cm/sec

cont'd ...

E. Be able to apply Logarithmic Differentiation.

Very useful, particularly when x or a function of x appears as a power.

e.g. Differentiate $y = (\sin x)^x$

=> $\ln y = \ln(\sin x)^x = x\ln(\sin x)$

=> $\frac{1}{y}\frac{dy}{dx} = x\frac{\cos x}{\sin x} + \ln(\sin x)$

=> $\frac{dy}{dx} = y(x\cot x + \ln(\sin x)) = (\sin x)^x(x\cot x + \ln(\sin x))$.

F. Be able to apply Parametric Differentiation.

Used when x and y are defined independently in terms of a 3rd variable (e.g. t).

e.g. If $x = 2t^2 + 3$ and $y = t^3$, find $\frac{dy}{dx}$.

=> $\frac{dx}{dt} = 4t$, $\frac{dy}{dt} = 3t^2$, => $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{3t^2}{4t} = \frac{3}{4}t$.

G. Be able to find the second derivative of a set of parametric equations.

Note that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$, but since $\frac{dy}{dx}$ is a function of t (not x), we proceed :-

=> $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \left(\frac{dt}{dx}\right) = \frac{d}{dt}\left(\frac{3}{4}t\right) \div \frac{dx}{dt} = \frac{3}{4} \div 4t = \frac{3}{16t}$

H. Know that the speed of a function, given in terms of a parameter t is :-

=> If $x = x(t)$ and $y = y(t)$, => speed $= |v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Mathematics 2

Unit 2 *Further Integration*

A. More Standard Integrals.

$f(x)$	$\int f(x)dx$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x + c$
$\frac{1}{1+x^2}$	$\tan^{-1}x + c$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

B. Be able to use Partial Fractions to help integrate functions.

e.g. Since $\frac{x^2-x+2}{(x-1)(x^2+1)} = \frac{1}{x-1} - \frac{1}{x^2+1}$, (Partial fractions)

=> $\int \frac{x^2-x+2}{(x-1)(x^2+1)}dx = \int \frac{1}{x-1}dx - \int \frac{1}{x^2+1}dx = \ln(x-1) - \tan^{-1}x + c$

C. Be able to use Integration by Parts

=> $$\int \left(u\frac{dv}{dx}\right)dx = uv - \int \left(v\frac{du}{dx}\right)dx$$

(a) For example use it to find $\int 2xe^x dx$

(b) It sometimes has to be applied more than once. e.g. $\int (x^2-1)e^x dx$

(c) It sometimes has to be used with a "dummy" variable

e.g. $\int \ln x dx = \int 1\ln x dx$ (let $u = \ln x$ and let $\frac{dv}{dx} = 1$ => $\frac{du}{dx} = \frac{1}{x}$ and $v = x$)

(d) Sometimes Integration by Parts "loops" back to the original function

e.g. $\int e^x \sin x dx$ loops round after two cycles to give :-

$$\int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

=> $2\int e^x \sin x dx = e^x \sin x - e^x \cos x$ => $\int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x)$

D. Be able to solve First Order Differential Equations which are Variable Separable.

If $\frac{dy}{dx}$ can be expressed as a product of a function of x and a function of y, :-

$$\Rightarrow \quad \frac{dy}{dx} = f(x)g(y)$$

$$\Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x)dx$$ and simply integrate both sides.

Also you should be able to find a <u>Particular Solution</u> given a set of values for x and y.

E. Be able to solve more complicated Problems by introducing Differential Equations which are Variable Separable.

e.g. The rate at which the number of people in a town (population N) catch a virus is proportional to the number who presently have the virus and the number who have not yet caught it. This is modelled by :-

$$\frac{dx}{dt} = kx(N - x)$$

$$\Rightarrow \quad \frac{dx}{x(N-x)} = kdt \quad \Rightarrow \quad \int \frac{dx}{x(N-x)}dx = \int ktdt$$ (use partial fractions)

Mathematics 2

Unit 3 *Complex Numbers*

A. Know that $\sqrt{-1} = i$. => $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, etc.

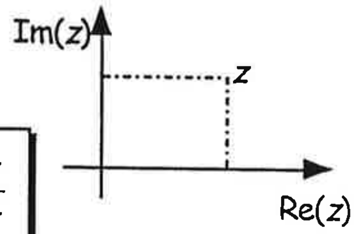
B. A Complex Number is of the form $z = x + iy$ (and its <u>conjugate</u> $\bar{z} = x - iy$).

C. Given $z_1 = a + ib$ and $z_2 = c + id$ be able to find (i) $z_1 + z_2$ and (ii) $z_1 - z_2$.

D. Be able to find $z_1 \times z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

E. Be able to divide using the "complex conjugate" => $\frac{z_1}{z_2} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)}$ etc.

F. Be able to represent $z = a + ib$ in an Argand Diagram.



G. Be able to change $z = x + iy$ to Polar Form as follows :-

=> $z = x + iy = r(\cos\theta + i\sin\theta)$ where $r = \sqrt{x^2 + y^2}$, $\tan\theta = \frac{y}{x}$

H. Know the terms Mod $z = |z| = r$. and argument z = arg $z = \theta$.

I. Be able to use De Moivre's Theorem as in $z = r(\cos\theta + i\sin\theta)$

=> $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all n (whole numbers)

(know that De Moivre's Theorem also works for negative and fractional values of n).

J. Fundamental Theorem of Algebra :- Every Polynomial equation of degree n has <u>exactly</u> n roots, some of them may be real, some repeated or some complex.

(know also if z is a root of a Polynomial equation, then $\bar{z}$ is also a root).

K. Be able to check if $z = 2 + 3i$ is a root of a polynomial equation by checking the factors

$(z - (2 + 3i))(z - (2 - 3i)) = ((z - 2) - 3i)((z - 2) + 3i) = z^2 - 4z + 13$ and

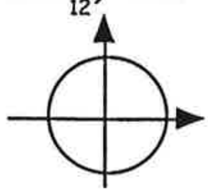
dividing the polynomial by $z^2 - 4z + 13$ to see if you get a zero remainder.

L. Be able to find (simple) roots of complex numbers e.g. given $z = 4\sqrt{2} + 4\sqrt{2}\, i$.

$z = 8(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ => $z^{1/3} = 8^{1/3}(\cos(\frac{1}{3} \times \frac{\pi}{4}) + i\sin(\frac{1}{3} \times \frac{\pi}{4})) = 2(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12})$ etc.

M. Recognise the Geometrical Interpretation of equations and inequalities.

e.g. $|z| = 4$ is represented by the circum of a circle as shown.



Mathematics 2

Unit 4 *Sequences and Series*

A. Recognise Arithmetic Sequences $a,\ (a+d),\ (a+2d),\ (a+3d),\ \ldots\ (a+(n-1)d).$

=> General term (nth term) => $u_n = (a+(n-1)d).$

B. Be able to find the sum of n terms of an Arithmetic Series :-

=> $S_n = a,\ (a+d)+(a+2d)+(a+3d)+\ \ldots + (a+(n-1)d) = \frac{n}{2}(2a+(n-1)d).$

C. Recognise Geometric Sequence $a,\ ar,\ ar^2,\ ar^3,\ \ldots\ ar^{(n-1)}.$

=> General term (nth term) => $u_n = ar^{(n-1)}.$

D. Be able to find the sum of n terms of an Geometric Series :-

=> $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{(n-1)} = \dfrac{a(1-r^n)}{(1-r)}$ or $\dfrac{a(r^n-1)}{(r-1)}.$

E. Recognise when a G.S. has a limit as the number of terms increases. (as $n \to \infty$)

=> $S_\infty = \dfrac{a}{1-r}$ as long as $-1 < r < 1$

F. Be able to use Simultaneous Equation Solving techniques to solve problems like :-

For an Arithmetic Sequence, the 3rd term is 8 and the 7th term is 32.

=> 3rd term = u_3 = $a + 2d = 8$

=> 7th term = u_7 = $a + 6d = 32$ etc.

G. Be able to expand :-

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots\ldots$$

$$\frac{1}{1+r} = 1 - r + r^2 - r^3 + \ldots\ldots$$

$$\frac{1}{x-y} = \frac{1}{x}\left(\frac{1}{1-\frac{y}{x}}\right) = \frac{1}{x}\left(1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 + \left(\frac{y}{x}\right)^3 + \ldots\ldots\right)$$

$$\frac{1}{2+3x} = \frac{1}{2}\left(\frac{1}{1+\frac{3x}{2}}\right) = \frac{1}{2}\left(1 - \left(\frac{3x}{2}\right) + \left(\frac{3x}{2}\right)^2 - \left(\frac{3x}{2}\right)^3 + \ldots\ldots\right)$$

H. Be able to use the $\sum$ (sigma) notation => $\sum_{k=1}^{k=n} k(k+1) = (1\times 2) + (2\times 3) + \ldots + (n(n+1)).$

I. Know (i) $\sum_{k=1}^{k=n} k = \frac{1}{2}n(n+1)$ (ii) $\sum_{k=1}^{k=n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ (iii) $\sum_{k=1}^{k=n} k^3 = \frac{1}{4}n^2(n+1)^2$

Mathematics 2

Unit 5 *Mathematical Proof*

A. Know the implication signs " => " " <= " and " <=> ".

B. Know about the Converse "If $p \Rightarrow q$ is a statement, $q \Rightarrow p$ is the converse".

C. Know that the Statement and / or the Converse may or may not be true.

D. If $p \Rightarrow q$ <u>and</u> $q \Rightarrow p$ then $p \Leftrightarrow q$. (they are <u>equivalent</u>.)

E. Know how to prove some statements **directly**. e.g. Proof of Pythagoras' Theorem.

F. Know how to use **Proof by Contradiction**, by

step 1 assume the opposite to the statement (the negative) is true.

step 2 show that using this assumption, <u>something</u> goes "wrong".

step 3 hence this means the original statement must have been true.

G. Know the principle of **Mathematical Induction** as a means of proof.

e.g. Prove that $\sum_{r=1}^{r=n} r = \frac{1}{2}n(n+1) \quad \forall\ n \in$ Natural Numbers

step 1 Prove it's true for $n = 1$ (show L.H.S. = R.H.S.)

step 2 Assume it's true for a particular value $n = k$

i.e. $\sum_{r=1}^{r=k} r = \frac{1}{2}k(k+1)$

and use this to prove it's also true for the next number, $n = k + 1$.

i.e $\sum_{r=1}^{r=k+1} r = \frac{1}{2}(k+1)(k+2)$ (somehow !)

this shows it is true $\forall\ n$ by Induction.

Mathematics 3

Mathematics 3

Unit 1 *Vectors*

A. Know all S5 Vector work, including :-

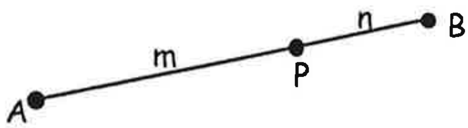
(a) Position Vector $\underline{p} = \overrightarrow{OP} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where P(x,y,z).

(b) Basic Laws of Vectors

add^n, subt^n	know to add (subtract) components
mult^n by scalar	$k\underline{u}$ = same dir^n as $\underline{u}$ (but x k)
commutative	$\underline{p}+\underline{q}=\underline{q}+\underline{p}$
associative	$\underline{p}+(\underline{q}+\underline{r})=(\underline{p}+\underline{q})+\underline{r}$
zero vector	$\underline{0}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
magnitude	$\underline{u}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ => $\lvert\underline{u}\rvert = \sqrt{x^2+y^2+z^2}$
unit vector	one whose magnitude $\lvert\underline{u}\rvert = 1$
$\underline{i}$, $\underline{j}$, $\underline{k}$ vectors	unit vectors parallel to axes.

(c) Section Formula

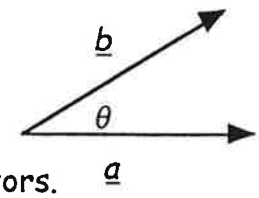
Given that point P divides AB in the ratio m : n, then position vector $\underline{p}$ is



$$\underline{p} = \frac{1}{m+n}\left(n\underline{a}+m\underline{b}\right)$$

(d) Scalar Product

$\underline{a}.\underline{b} = a_1b_1 + a_2b_2 + a_3b_3$
$= \lvert\underline{a}\rvert\lvert\underline{b}\rvert\cos\theta$



use it to calculate angles between vectors.

=> $\cos\theta = \dfrac{\underline{a}.\underline{b}}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert} = \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\lvert\underline{a}\rvert\lvert\underline{b}\rvert}$

=> know that $\underline{a}.\underline{b} = 0$ <=> $\underline{a}$ is perpendicular to $\underline{b}$

B. Be able to calculate the **vector product** (or cross product).

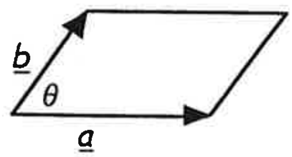
Given $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, then:-

=> $$\underline{a} \times \underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \underline{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \underline{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

C. Know that $\underline{a} \times \underline{b}$ produces a vector <u>perpendicular</u> to the plane containing $\underline{a}$ and $\underline{b}$.

Know also that $|\underline{a} \times \underline{b}| = |\underline{a}| \times |\underline{b}| \times \sin\theta$,

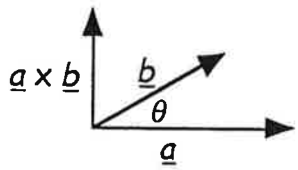
which is the area of the parallelogram formed from $\underline{a}$ and $\underline{b}$.



D. Know that if $\underline{a} \times \underline{b} = \underline{0}$ => $\underline{a}$ is parallel to $\underline{b}$ => $\underline{b} = k\underline{a}$.

Know also that $\underline{a} \times \underline{b} = -(\underline{b} \times \underline{a})$

Know also that $(\underline{a}, \underline{b}, (\underline{a} \times \underline{b}))$ forms a "right handed system"



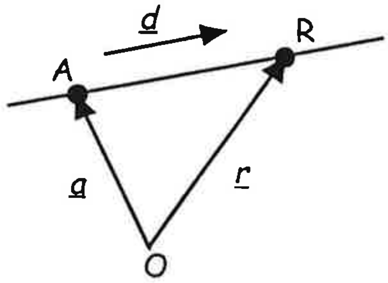
Know that $\underline{a} \times (\underline{b} + \underline{c}) = (\underline{a} \times \underline{b}) + (\underline{a} \times \underline{c})$ (the distributive property)

Know also that

$\times$	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{i}$	$\underline{0}$	$\underline{k}$	$-\underline{j}$
$\underline{j}$	$-\underline{k}$	$\underline{0}$	$\underline{i}$
$\underline{k}$	$\underline{j}$	$-\underline{i}$	$\underline{0}$

E. <u>The Equations of a Line</u> :-

(i) Vector Form :- $\underline{r} = \underline{a} + t\underline{d}$.



(ii) Parametric Form
$$x = a + tl$$
$$y = b + tm$$
$$z = c + tn$$

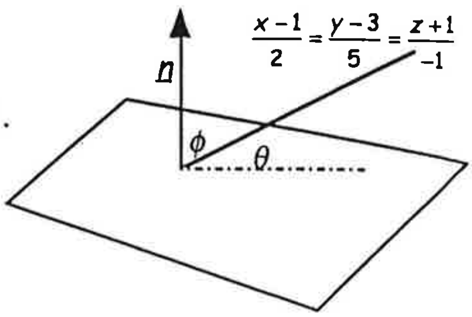
(iii) Symmetric Form (Cartesian) $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ $(= t)$

(iv) Know that to find a line you need to know <u>2 things</u> :- a point on the line and the line's direction ratios. (the l, m and n)

I. Be able to find the angles between a line and a plane.

Step 1 Find the angle ϕ between the line and the normal.

Step 2 The angle we want, θ, is simply $(\frac{\pi}{2} - \phi)$.



J. Be able to find the point of intersection of 2 lines.

(i) Lines will not meet if parallel

e.g. $\frac{x}{3} = \frac{y+2}{-2} = \frac{z-5}{4}$ and $\frac{x+2}{3} = \frac{y}{-2} = \frac{z-3}{4}$ are parallel.

(ii) If not parallel they might (or might not) meet :-

e.g. to check if $\frac{x+1}{3} = \frac{y+2}{-2} = \frac{z-5}{4}$ and $\frac{x+2}{2} = \frac{y+5}{1} = \frac{z-3}{-2}$ meet

look at parametric form $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

solve the 1st two simultaneous equations $3t - 1 = 2s - 2$
and $-2t - 2 = 1s - 5$

and check if the solution satisfies the 3rd $4t + 5 = -2s + 3$

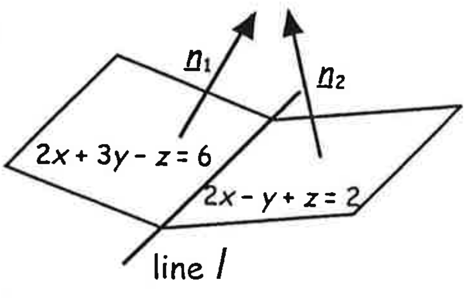
If it does the lines meet (sub the value for t (or s) in). If not, they don't meet.

K. Be able to find the equation of the line of intersection of two planes

If the two planes are parallel ($\underline{n}_1 = \underline{n}_2$) they won't meet.

If they are not parallel, find the line as follows :-

e.g. $2x + 3y - z = 6$ and $3x - y + z = 2$



Step 1 $\underline{n}_1 \times \underline{n}_2 = \underline{a}$ gives the direction of the line.

Step 2 set x (or y or z) = 0 (or any number) and solve the equations $3y - z = 6$ and $-y + z = 2$.
=> $y = 4, z = 6, x = 0$ => P(0, 4, 6) is a point on the line of intersection.

Step 3 use the direction $\underline{a}$ and the point P to find the line.

L. To find where three planes meet :-

They either meet at a line (see K, above)

They do NOT meet at all.

They meet at a single point (use Gaussian elimination techniques)

Mathematics 3

Unit 2 *Matrices*

A. Be able to apply earlier work on Matrices, (see Maths 1, unit 5)

B. Know the Identity matrix (for multiplication) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

C. Given a 2 x 2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, be able to find its inverse $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

D. Know that if det$A$ = | A | = 0, no inverse exists. A is said to be singular.

E. Be able to add, subtract and multiply matrices when appropriate.

F. Find the determinant of a 3 x 3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

=> $\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

G. Know that $(AB)^{-1} = B^{-1}A^{-1}$ and $(AB)^T = B^T A^T$ (or $(AB)' = B'A'$)

H. Use matrix inverses to solve 2 x 2 systems of equations in 2 variables.

I. Given a square matrix A, be able to find values p and q such that $A^2 = pA + qI$ etc.

J. Be able to find the inverse of a 3 x 3 matrix (if invertible) using Gaussian elimination and elementary row operations.

K. Know that certain 2 x 2 matrices represent geometric transformations such as reflections, rotations and dilatations (dilations). e.g.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ - reflection over x-axis

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ - reflection over y-axis

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ - reflection through origin

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ - reflection over line $y = x$.

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ - 90° rotation clockwise around O

$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ dilatation, centre O, scale factor 4

Mathematics 3

Unit 3 *Further Sequences and Series.*

A. Know how to find the Maclaurin Power Series Expansion for a given function $f(x)$.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + ...$$

(Know that it might converge $\forall x$, diverge or converge for a limited range of x.)

B. Know, (to save time),

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} + ...$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

$$\ln(1 - x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + ...$$

C. Iteration - be able to "home in" on the solution of an equation between two values $x = a$ and $x = a + 1$, using technique learned in fifth year.

D. Be able to solve $y = f(x) = 0$, by using a rearrangement $x = g(x)$ and the recurrence relation $x_{n+1} = g(x_n)$ to "home in" on the solution.

(Know that the starting value, a, will converge to the solution iff $|g'(a)| < 1$)
(If $|g'(a)| \geq 1$, there will be a divergence and the solution at a will not be found).

E. Recognise "staircase" and "cobweb" diagrams.

Mathematics 3

Unit 4 *Further Differential Equations.*

A. Be able to solve First Order Linear Differential Equations (F.O.L.D.E.) like

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad \text{-------------} \; 1$$

by using the **integrating factor** $\boxed{\mu(x) = e^{\int P(x)dx}}$

The solution of 1 becomes $\boxed{y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx}$

B. Be able to solve Second Order Differential Equations

(a) Homogeneous $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ by using the "big D" method.

The "Auxiliary Equation" is $aD^2 + bD + c = 0$ and depends on the type of solutions

(i) If 2 real answers, $D = \alpha$ and β, the solution is $y = Ae^{\alpha x} + Be^{\beta x}$

(ii) If only 1 real answer, $D = \alpha$, the solution is $y = (Ax + B)e^{\alpha x}$

(iii) If 2 complex solutions $D = \alpha \pm i\beta$, solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

This is called the "**Complementary Function**" (the C.F.)

(b) Non - Homogeneous $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where $f(x)$ is a polynomial, trigonometric or exponential function.

Step 1 solve the corresponding homogeneous case to obtain the **Complementary Function** (C.F.)

Step 2 Try to find the **Particular Integral** (the P.I.) by attempting a solution of the "same type" as $f(x)$.

Step 3 The General solution of the non-homogeneous differential equation is

General Solution = Complementary Function + Particular Integral.

(c) Know that if the complementary function contains terms which appear in the right side, $f(x)$, (e.g. $y = Ae^{2x} + Be^{3x}$ and $f(x) = 3e^{2x}$), then try as a possible particular integral $y = axe^{2x}$, instead of $y = ae^{2x}$.

Mathematics 3

Unit 5 *Elementary Number Theory.*

A. Be able to prove or disprove statements using earlier work. (See Maths 2. unit 5).

B. Be able to apply Proof by Induction using the $\sum$ notation.

C. Know the Division Algorithm :-

Given 2 whole numbers, a and b, there exists <u>unique</u> integers q and r, such that

if $a > b$ => $a = qb + r$

q is called the "quotient" and r is called the "remainder".

D. Be able to apply the Euclidian Algorithm to find the g.c.d. of any two integers.

e.g. to find the g.c.d. of 136 and 221, proceed as follows

$$221 = 1 \times \underline{136} + \underline{85}$$
$$136 = 1 \times \underline{85} + \underline{51}$$
$$85 = 1 \times \underline{51} + \underline{34}$$
$$51 = 1 \times \underline{34} + \underline{17}$$
$$* \; 34 = 2 \times \underline{17} + 0 \quad \Rightarrow \quad \text{g.c.d.}(136,221) = \underline{17}.$$

E. Be able to express the g.c.d. of $(a, b) = d$ as a linear multiple of a and b.

i.e. be able to find integers x and y, such that $xa + yb = d$. by working backwards from *.

F. Be able to express any base 10 number in another base and vice-versa.

e.g. 218_{10}, expressed in base 6 becomes :-

6	218	
6	36	r 2
6	6	r 0
6	1	r 0
6	0	r 1

(remainders read upwards) => 1002_6

e.g. $1002_6 = 1 \times 6^3 + 0 \times 6^2 + 0 \times 6^1 + 2 = 218_{10}$.