Section 4

Question Bank

Differentiate with respect to x

(a) $y = x^2 \tan^{-1} x$, **2**

(b) $y = \dfrac{\ln x}{x^3 + 1}$, $\quad x > 0$. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	2.1	2		2.1.1	1.2.4	1996 SY1 Q1
(b)	3	1.2	3		1.2.4	1.2.5	

(a)

$$y = x^2 \tan^{-1} x$$

$$\frac{dy}{dx} = 2x \tan^{-1} x + \frac{x^2}{1 + x^2}$$

1 for the product rule

1 for derivative of $\tan^{-1} x$

(b)

$$y = \frac{\ln x}{x^3 + 1}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}(x^3 + 1) - 3x^2 \ln x}{(x^3 + 1)^2}$$

1 for quotient rule

1 for handling $\ln x$

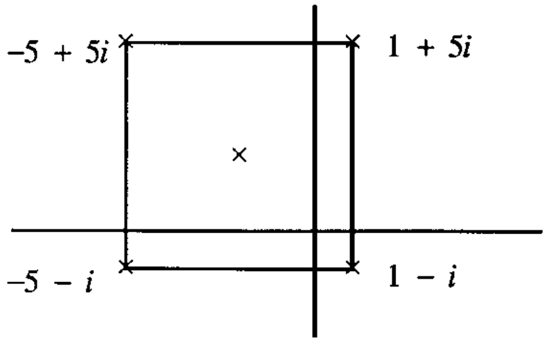
1 for derivative of denominator

$$= \frac{(x^3 + 1) - 3x^3 \ln x}{x(x^3 + 1)^2}$$

The point A represents $-5 + 5i$ on an Argand diagram and $ABCD$ is a square with centre $-2 + 2i$. Find the complex numbers represented by the points B, C and D, giving your answers in the form $x + iy$. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	2.3	4		2.3	2.3.5	1996 SY1 Q2

A is $-5 + 5i$, centre $-2 + 2i$.



1 for a diagram or other approach

$1 + 5i$, $1 - i$, $-5 - i$. **1, 1, 1**

Use Gaussian elimination to solve the following system of equations.

$$\begin{aligned} x \quad\quad - z &= 2 \\ 2y - 3z &= 6 \\ 2x + y + z &= 1 \end{aligned}$$ **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.5	5		1.5.4		1996 SY1 Q5

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 2 & 1 & 1 & 1 \end{array}\right)$$ **1**

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 1 & 3 & -3 \end{array}\right)$$ **1**

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -3 & 6 \\ 0 & 0 & 4{\cdot}5 & -6 \end{array}\right)$$ **1**

$z = -\frac{4}{3};$ **1**

$y = 1; x = \frac{2}{3}$ **1**

I have just received a letter from Scottish Condensed Books PLC telling me that I have won a prize in their contest. They tell me that I have won an income for the rest of my life and have arranged for the prize to be paid as follows.

> At the **end** of the first year, I will receive £10 000; at the end of the second year I will get £12 000; at the end of the third year £14 000 and so on, with the amount increasing by £2000 each year.

What will my total prize money amount to at the end of the n^{th} year? **3**

How many years will I have to live before the accumulated prize money exceeds £250 000? **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.4	3		2.4.2		1996 SY1 Q7
	2	2.4	2		2.4.2		

An arithmetic series with $a = 10000$ and $d = 2000$. **1**

$$S_n = 10000 + 12000 + 14000 + \dots$$

$$= \frac{n}{2}\{2a + (n-1)d\} \quad \mathbf{1}$$

$$= \frac{n}{2}\{20000 + (n-1)2000\} \quad \mathbf{1}$$

$$= n\{10000 + 1000n - 1000\}$$

$$= 1000n(9+n).$$

Note that the formula need not be simplified to obtain the third mark.

$$1000n(9+n) > 250000 \quad \textbf{1 for an initial inequality}$$

$$n^2 + 9n > 250$$

$$(n + 4\tfrac{1}{2})^2 > 250 + 20\tfrac{1}{4} = 270\tfrac{1}{4}$$

$$n + 4\tfrac{1}{2} > 16.439\ldots$$

$$n > 16.439 - 4.5 = 11.939$$

It takes 12 years. **1**

(a) By using the substitution $u = 2\sin x$, or otherwise, evaluate the definite integral

$$\int_0^{\pi/6} \frac{\cos x}{1 + 4\sin^2 x}\, dx.$$ **4**

(b) Use integration by parts to find

$$\int x^2 \ln x\, dx.$$ **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.3	4		1.3.4		1996 SY1 Q8
(b)	3	2.2	3		2.2.2		

(a) $u = 2\sin x;\ du = 2\cos x\, dx$ **1**

$x = 0 \Rightarrow u = 0; x = \frac{\pi}{6} \Rightarrow u = 1$ **1**

$$\int_0^{\frac{\pi}{6}} \frac{\cos x}{1 + 4\sin^2 x}\, dx = \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{1}{1 + u^2}\, du$$

$$= \tfrac{1}{2}\left[\tan^{-1} u\right]_0^1$$ **1**

$$= \frac{\pi}{8}.$$ **1**

(b) $$\int x^2 \ln x\, dx = \ln x \int x^2\, dx - \int \tfrac{1}{x}\left(\tfrac{1}{3}x^3\right) dx$$ **1**

$$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c.$$ **1 for first term**

1 for second term

Solve the equation

$$z^2 + \sqrt{8}z + 4 = 0$$

for the complex number z.

Give the modulus and argument of each of the roots and illustrate them on an Argand diagram. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.3	5		2.3.11	2.3.5	1996 SY1 Q10

$$z^2 + \sqrt{8}z + 4 = 0$$

$$z = \frac{-\sqrt{8} \pm \sqrt{8 - 16}}{2}$$ **1 for using formula**

$$= \frac{-\sqrt{8} \pm \sqrt{8}i}{2}$$ **1**

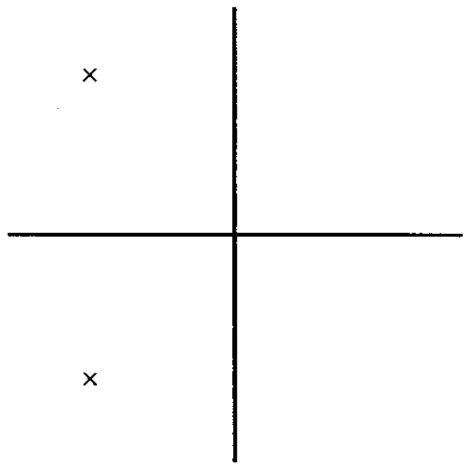
$$= \sqrt{2}(-1 \pm i)$$

$$z_1 = \sqrt{2}(-1 + i)$$

$$\Rightarrow |z_1| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2 \text{ and } \arg z_1 = \tan^{-1}\left(\frac{\sqrt{2}}{-\sqrt{2}}\right) = \frac{3\pi}{4}$$ **1**

$$z_2 = \sqrt{2}(-1 - i)$$

$$\Rightarrow |z_2| = \sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2 \text{ and } \arg z_2 = \tan^{-1}\left(\frac{-\sqrt{2}}{-\sqrt{2}}\right) = \frac{5\pi}{4}$$ **1**



1

Let f be the function given by

$$f(x) = 6x^3 - 5x - 1.$$

Find algebraically the values of x for which the slope of this function is between -3 and 3. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.2	5		1.2.8		1996 SY1 Q12

$$f(x) = 6x^3 - 5x - 1.$$

$$f'(x) = 18x^2 - 5$$

$$18x^2 - 5 < 3$$ **1**

$$18x^2 < 8$$

$$x^2 < \frac{4}{9}$$

$$-\frac{2}{3} < x < \frac{2}{3}$$ **1**

$$18x^2 - 5 > -3$$ **1**

$$18x^2 > 2$$

$$x^2 > \frac{1}{9}$$

$$x > \frac{1}{3} \text{ or } x < -\frac{1}{3}$$ **1**

Combining these

$$-\tfrac{2}{3} < x < -\tfrac{1}{3} \text{ or } \tfrac{1}{3} < x < \tfrac{2}{3}.$$ **1**

When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water.
This can be represented by the differential equation

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}, \qquad h \geqslant 0,$$

where h is the depth of the water (in metres) and t is the time (in minutes) elapsed since the valve was opened.

(a) Express h as a function of t. **4**

(b) Find the solution of the equation given that the pool was initially 4 m deep. **2**

(c) The next time the pool had to be drained the water was initially 9 m deep. How long will it take to drain the pool on this occasion? **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.2	4		2.2.5		1996 SY1 Q13
(b)	2	2.2	2		2.2.5		
(c)	3	2.2		3	2.2.5		

(a)

$$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}$$

$$\int h^{-½}\,dh = \int -\frac{1}{10}\,dt \qquad \mathbf{1}$$

$$\frac{h^{½}}{½} = -\frac{t}{10} + c \qquad \mathbf{1}$$

$$\sqrt{h} = \frac{c}{2} - \frac{t}{20} \qquad \mathbf{1}$$

$$h = \left(\frac{c}{2} - \frac{t}{20}\right)^2 \qquad \mathbf{1}$$

(b) $t = 0,\ h = 4 \Rightarrow 4 = (\frac{c}{2})^2 \Rightarrow c = 4$ **1**

$\therefore \quad h = (2 - \frac{t}{20})^2$ **1**

(c) Using $h = (\frac{c}{2} - \frac{t}{20})^2$ again

$t = 0,\ h = 9 \Rightarrow 9 = (\frac{c}{2})^2 \Rightarrow c = 6$ **1**

$\therefore \quad h = (3 - \frac{t}{20})^2$ **1**

$h = 0 \Rightarrow t = 60$ **1**

The pool would take 1 hour to empty.

(a) (i) Show that

$$\frac{3}{2}\int_0^a \sqrt{x+1}\,dx = (a+1)^{\frac{3}{2}} - 1 \qquad \mathbf{2}$$

(ii) Hence find $\int_0^1 \sqrt{x+1}\,dx$ correct to 3 decimal places. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)(i)	2	1.3	2		1.3.2		1996 SY1 Q14
(a)(ii)	2	1.3	2		1.3.2		

(a) (i)

$$\frac{3}{2}\int_0^a \sqrt{x+1}\,dx = \left[(x+1)^{\frac{3}{2}}\right]_0^a \qquad \mathbf{1}$$

$$= (a+1)^{\frac{3}{2}} - 1 \qquad \mathbf{1}$$

(ii)

$$\int_0^1 \sqrt{x+1}\,dx = \frac{2}{3}\left[(1+1)^{\frac{3}{2}} - 1\right] \qquad \mathbf{1}$$

$$= \frac{2}{3}(2\sqrt{2} - 1) = 1.219 \text{ to 3 dps} \qquad \mathbf{1}$$

(a) Find a real root of the cubic polynomial

$$c(x) = x^3 - x^2 - x - 2$$

and hence factorise it as a product of a linear term $l(x)$ and a quadratic term $q(x)$. **3**

(b) Show that $c(x)$ cannot be written as a product of three real linear factors. **1**

(c) Use your factorisation to find values of A, B and C such that

$$\frac{5x + 4}{x^3 - x^2 - x - 2} = \frac{A}{l(x)} + \frac{Bx + C}{q(x)}.$$

Hence obtain the indefinite integral

$$\int \frac{5x + 4}{x^3 - x^2 - x - 2}\, dx.$$ **7**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.1	3				1996 SY1 Q15
(b)	1	1.1	1				
(c)	7	2.2		7	2.2.1		

(a) $c(x) = x^3 - x^2 - x - 2$

$c(2) = 8 - 4 - 2 - 2 = 0$ **1**

$\therefore c(x) = (x - 2)(x^2 + x + 1)$ **1 for division,**
1 for stating $c(x)$

(b) The discriminant of $x^2 + x + 1$ is $1 - 4 < 0$ so no real factors. **1**

(c) $$\frac{5x + 4}{x^3 - x^2 - x - 2} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + x + 1}.$$ **1**

$5x + 4 = A(x^2 + x + 1) + (Bx + C)(x - 2)$ **1**

$x = 2; \quad 14 = 7A \Rightarrow A = 2$ **1**

$x = 0; \quad 4 = 2 + C(-2) \Rightarrow C = -1$ **1**

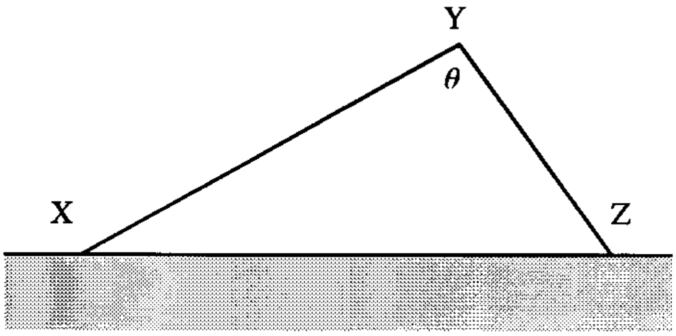
$x = 1; \quad 9 = 6 + (B + 1)(-1) \Rightarrow B = -2$ **1**

$$\int \frac{5x + 4}{x^3 - x^2 - x - 2}\, dx = \int \frac{2}{x - 2} - \frac{2x + 1}{x^2 + x + 1}\, dx$$ **1**

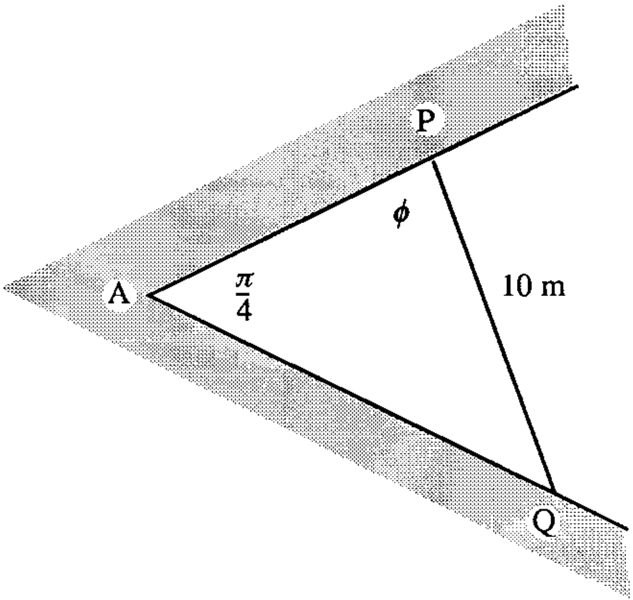
$$= \ln(x - 2) - \ln(x^2 + x + 1) + c$$ **1**

$$= \ln\frac{x - 2}{x^2 + x + 1} + c.$$

(a) Two straight pieces of fencing XY and YZ, linked at Y, are used together with a straight wall to make an enclosure XYZ as shown. Given that XY has length a metres and YZ length b metres, write down an expression for the area of XYZ and find the value of the angle θ which maximises this area. **3**



(b) Two straight walls meet at A at an angle $\frac{\pi}{4}$. A straight piece of fencing PQ of length 10 metres is used to create an enclosure APQ as shown.



(i) Show that the area APQ is given by

$$50\sqrt{2}\,\sin\phi\,\sin\left(\frac{3\pi}{4} - \phi\right).$$ **4**

(ii) If the angle ϕ is varied, using differentiation or otherwise, find the value of ϕ which maximises the area APQ. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.2	3		1.2.8		1996 SY1 Q16
(b)(i)	4	1.2	4		1.2.8		
(b)(ii)	4	1.2		4	1.2.8		

(a) Area $A_1(\theta) = \frac{1}{2}ab \sin\theta$ **1**

$A_2'(\theta) = \frac{1}{2}ab\cos\theta = 0$ when $\theta = \frac{\pi}{2}$. **1**

$A_2''(\theta) = -\frac{1}{2}ab\sin\theta;\ A_2''(\frac{\pi}{2}) = -\frac{1}{2}ab < 0$ **1**

$\therefore$ the area is maximum when $\theta = \frac{\pi}{2}$.

(b) (i)

$$\angle AQP = \frac{3\pi}{4} - \phi$$ **1**

$$\frac{AQ}{\sin\phi} = \frac{10}{\sin\frac{\pi}{4}}$$ **1**

$$\Rightarrow AQ = 10\sqrt{2}\sin\phi$$ **1**

Area of APQ, $$A_2(\phi) = \frac{1}{2} \times 10 \times 10\sqrt{2}\sin\phi\sin\left(\frac{3\pi}{4} - \phi\right)$$ **1**

$$= 50\sqrt{2}\sin\phi\sin\left(\frac{3\pi}{4} - \phi\right)$$

(ii)

$$A_2'(\phi) = 50\sqrt{2}\left[\cos\phi\sin\left(\frac{3\pi}{4} - \phi\right) - \sin\phi\cos\left(\frac{3\pi}{4} - \phi\right)\right]$$ **1**

$$= 50\sqrt{2}\sin\left(\frac{3\pi}{4} - 2\phi\right) = 0 \text{ at stationary values}$$ **1**

$$\therefore \frac{3\pi}{4} - 2\phi = 0,\ \pi,\ 2\pi,\ \ldots$$

Likely to be $\phi = \frac{1}{2}(\frac{3\pi}{4}) = \frac{3\pi}{8}$. **1**

$$A_2''(\phi) = -100\sqrt{2}\cos\left(\frac{3\pi}{4} - 2\phi\right)$$

$$A_2''\left(\frac{3\pi}{8}\right) = -100\sqrt{2}\cos\left(\frac{3\pi}{4} - \frac{3\pi}{4}\right) < 0$$ **1**

i.e. the area is maximum when $\phi = \frac{3\pi}{8}$.

Use the Euclidean Algorithm to find integers x, y such that

$$83x + 239y = 1.$$ **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	3.5	2		3.5.10		1996 SY2 Q1
	2	3.5		2	3.5.11		

$$239 = 2 \times 83 + 73$$ **1**

$$83 = 1 \times 73 + 10$$

$$10 = 3 \times 3 + 1$$ **1**

Thus

$$1 = 10 - 3(73 - 7 \times 10)$$ **1**

$$= 22 \times 10 - 3 \times 73$$

$$= 22(83 - 73) - 3 \times 73$$

$$= 22 \times 83 - 25(239 - 2 \times 83)$$

$$= 72 \times 83 - 25 \times 239$$ **1**

Use induction to prove that, for all positive integers n,

$$\sum_{r=1}^{n} \frac{r}{2^r} = 2 - \frac{(n+2)}{2^n}.$$ **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	3.5			3.5.6		1996 SY2 Q3

Check $n = 1$: LHS $= \frac{1}{2}$; RHS $= 2 - \frac{3}{2} = \frac{1}{2}$. **1**

Suppose true for $n = k$, i.e.

$$\sum_{r=1}^{k} \frac{r}{2^r} = 2 - \frac{k+2}{2^k}$$ **1**

Then

$$\sum_{r=1}^{k+1} \frac{r}{2^r} = \sum_{r=1}^{k} \frac{r}{2^r} + \frac{k+1}{2^{k+1}}$$ **1**

$$= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}}$$

$$= 2 - \frac{1}{2^{k+1}}(2k + 4 - k - 1)$$ **1**

$$= 2 - \frac{k+3}{2^{k+1}}$$ **1**

$$= 2 - \frac{(k+1)+2}{2^{k+1}}$$

So the result is true for $n = 1$, and is true for $n = k + 1$ whenever it is true for $n = k$. So it is true for all $n \geqslant 1$. **1**

The square $n \times n$ matrix A satisfies the equation

$$A^2 = 5A - 6I$$

where I is the $n \times n$ identity matrix. Show that A is invertible and express A^{-1} in the form $pA + qI$. **2**

Obtain a similar expression for A^3. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	3.2	2		3.2.2		1996 SY2 Q4
	2	3.2		2	3.2.2		

$$A^2 = 5A - 6I, \text{ so}$$

$$6I = A(5I - A) \qquad \mathbf{1}$$

$$I = A\left[\tfrac{1}{6}(5I - A)\right]$$

So A^{-1} exists and equals $\frac{1}{6}(5I - A)$. **1**

[An answer which uses A^{-1} in the early stages can only gain 1 mark.]

Thus

$$A^3 = A(5A - 6I) \qquad \mathbf{1}$$

$$= 5A^2 - 6A = 5(5A - 6I) - 6A$$

$$= 19A - 30I \qquad \mathbf{1}$$

(a) Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ where $a, b, c, d \in \mathbf{R}$.

Find AB and evaluate the determinants $\det A$, $\det B$ and $\det AB$.

By using the fact that $\det AB = \det A \,.\, \det B$, deduce the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$ **4**

(b) Write each of the prime numbers 17 and 41 as the sum of the squares of two positive integers.

Hence use the result from (i) to express 697 as $p^2 + q^2$ and as $r^2 + s^2$ where p, q, r and s are distinct positive integers. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	3.2	4		3.2.4		1996 SY2 Q7
(b)	4	2.5	4				

(a)

$$AB = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$ **1**

$$\det A = a^2 + b^2; \det B = c^2 + d^2$$ **1**

$$\det AB = (ac - bd)^2 - (-ad - bc)(ad + bc)$$ **1**

$$= (ac - bd)^2 + (ad + bc)^2$$ **1**

Hence

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2$$ **1**

(b)

$$17 = 4^2 + 1^2, \qquad 41 = 4^2 + 5^2$$ **1**

Thus

$$697 = 17 \times 41$$

$$= (4^2 + 1^2) + (4^2 + 5^2)$$ **1**

$$= (4 \times 4 - 1 \times 5)^2 + (4 \times 5 + 1 \times 4)^2$$

$$= 11^2 + 24^2$$ **1**

and also

$$697 = (4^2 + 1^2)(5^2 + 4^2) = 16^2 + 21^2$$ **1**

(a) Let $\mathbf{u} = \mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find $\mathbf{u.v}$ and $\mathbf{u} \times \mathbf{v}$. **2**

(b) Three planes π_1, π_2 and π_3 are given by the equations

$$\pi_1 : x - 4y - z = 3$$

$$\pi_2 : 2x - 2y + z = 6$$

$$\pi_3 : 3x - 11y - 2z = 10.$$

(i) Find the acute angle between the planes π_1 and π_2. **2**

(ii) By using Gaussian elimination, or otherwise, show that the planes π_1, π_2 and π_3 intersect in a point Q, and obtain the coordinates of Q. **4**

(iii) Find an equation for the line L in which π_1 and π_2 intersect, and the point R in which L intersects the xy-plane. **4**

(c) Three non-zero vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \quad \text{and} \quad \mathbf{b} \times \mathbf{c} = \mathbf{a}.$$

Explain briefly why $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ must be mutually perpendicular and why $\mathbf{b}$ must be a unit vector. **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	3.1	2		3.1.4		1996 SY2 Q8
(b)(i)	2	3.1		2	3.1.8		
(b)(ii)	4	3.1	4		3.1.9	1.5.4	
(b)(iii)	4	3.1		4	3.1.7		
(c)	3	3.1	3		3.1.1		

(a) $\mathbf{u}.\mathbf{v} = 2 + 8 - 1 = 9$ **1**

$\mathbf{u} \times \mathbf{v} = -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ **1**

(b) (i) The normals to the planes are **u** and **v**, **1**
so the required angle is θ where

$$9 = \sqrt{18}\sqrt{9}\cos\theta$$

i.e. $\cos\theta = \frac{1}{\sqrt{2}}$, so $\theta = \frac{\pi}{4}$ (or 45°) **1**

(ii)

$$\left(\begin{array}{ccc|c} 1 & -4 & -1 & 3 \\ 2 & -2 & 1 & 6 \\ 3 & -11 & -2 & 10 \end{array}\right)$$ **1**

$$\left(\begin{array}{ccc|c} 1 & -4 & -1 & 3 \\ 0 & 6 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right)$$ **1**

$$\left(\begin{array}{ccc|c} 1 & -4 & -1 & 3 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \end{array}\right)$$ **1**

The planes do intersect with $\frac{1}{2}z = 1$, i.e. $z = 2$, $y = -1$, $x = -4 + 2 + 3$.
So Q is $(1, -1, 2)$. **1**

(iii) **Note that this is just one of many routes to one of many equations.**
L is in the direction of $\mathbf{u} \times \mathbf{v} = -6\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$, **1**
and L passes through Q, so an equation is

$$\frac{x-1}{-6} = \frac{y+1}{-3} = \frac{z-2}{6} (= s)$$ **1**

L meets the xy-plane where $z = 0$,
i.e. where $s = -\frac{1}{3}$. **1**
So $x = 3$, $y = 0$, i.e. R is $(3, 0, 0)$ **1**

(c) $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, so **a** is perpendicular to **c** and
b is perpendicular to **c**
and
$\mathbf{b} \times \mathbf{c} = \mathbf{a}$, so **a** is perpendicular to **b** **1**

Since all three vectors are mutually perpendicular
$\mathbf{a} \times \mathbf{b} = \mathbf{c} \Rightarrow ab = c$, where $a = |\mathbf{a}|$, etc.
$\mathbf{b} \times \mathbf{c} = \mathbf{a} \Rightarrow bc = a$ **1**
Thus $(bc)b = c \Rightarrow b^2 = 1 \Rightarrow b = 1$ and the rest follow. **1**

(a) Prove that $\sqrt{2}$ is irrational. **4**

(b) (i) Show that if

$$A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

then

$$x_1^2 - 2y_1^2 = x_0^2 - 2y_0^2.$$ **3**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	3.5	4		3.5.4		1996 SY2 Q10
(b)(i)	3	3.2	3		3.2.9		

(a) Suppose that $\sqrt{2} = \frac{m}{n}$ where the integers m, n have **no common factor.** **1**

Then $m^2 = 2n^2$. **1**

Thus m is even, so $m = 2u$ for some integer u. **1**

Thus $4u^2 = 2n^2$ i.e. $n^2 = 2u^2$, so n is also even, **contradicting** the assumption. **1**

(b) (i) $x_1 = 3x_0 + 4y_0$ **and** $y_1 = 2x_0 + 3y_0$ **1**

so

$$x_1^2 - 2y_1^2 = 9x_0^2 + 24x_0y_0 + 16y_0^2 - 8x_0^2 - 24x_0y_0 - 18y_0^2$$ **1**

$$= x_0^2 - 2y_0^2$$ **1**

Differentiate the following functions with respect to x.

(a) $y = x^3e^{-x^2}$, **2**

(b) $f(x) = \tan^{-1}(\sqrt{x-1}), \quad x > 1$ **2**

(c) $f(x) = \dfrac{x^2}{\cos x}, \quad \dfrac{-\pi}{2} < x < \dfrac{\pi}{2}$. **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	1.2	2		1.2.4	1.2.5	1997 SY1 Q1
(b)	2	2.1	2		2.1.1		
(c)	2	1.2	2		1.2.4		

(a) $y = x^3e^{-x^2} \Rightarrow$

$$\frac{dy}{dx} = 3x^2e^{-x^2} + x^3e^{-x^2}(-2x)$$ **1 for product rule**

1 for $\dfrac{d}{dx}\left(e^{-x^2}\right) = (-2x)e^{-x^2}$

$$= 3x^2e^{-x^2} - 2x^4e^{-x^2}$$

(b) $f(x) = \tan^{-1}(\sqrt{x-1}) = \tan^{-1}(x-1)^{½}$

$$f'(x) = \frac{1}{1+(x-1)} \cdot ½(x-1)^{-½}$$ **1 for $\dfrac{1}{1+(x-1)}$**

1 for $½(x-1)^{-½}$

$$= \frac{1}{2x\sqrt{x-1}}$$

(c) $f(x) = \dfrac{x^2}{\cos x}$

$$f'(x) = \frac{2x\cos x - x^2(-\sin x)}{\cos^2 x}$$ **1 for qoutient rule**

1 for accuracy

$$= \frac{2x\cos x + x^2\sin x}{\cos^2 x}.$$

By means of the substitution $u = x^2 - 8$, find

$$\int_3^4 x^3(x^2 - 8)^{\frac{1}{3}}\,dx.$$ 5

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.3	5		1.3.4		1997 SY1 Q5

$$u = x^2 - 8 \Rightarrow du = 2x\,dx \Rightarrow dx = ½du$$ **1**

$$x = 3 \Rightarrow u = 1 \text{ and } x = 4 \Rightarrow u = 8$$ **1**

$$\int_3^4 x^3(x^2-8)^{1/3}dx = \int_1^8 (u+8)(u)^{1/3}½du$$ **1**

$$= ½\int_1^8 \left(u^{4/3} + 8u^{1/3}\right)du$$

$$= ½\left[\frac{3}{7}u^{7/3} + 6u^{4/3}\right]_1^8$$ **1**

$$= ½\left[\frac{384}{7} + 96\right] - ½\left[\frac{3}{7} + 6\right]$$ **1**

$$= ½\left[54\frac{3}{7} + 90\right] = 72\frac{3}{14}$$

Let A be the matrix $\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

Show that $A^2 - 7A = cI$ where c is a real number and I is the 2×2 identity matrix. **3**

By considering this equation in the form

$$A(A - 7I) = cI,$$

obtain the matrix B for which

$$AB = I.$$ **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	3.2	3		3.2.2	3.2.1	1997 SY1 Q6
	2	3.2		2	3.2.2		

$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$ so $A^2 = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 18 & 7 \\ 14 & 11 \end{pmatrix}$ **1**

Hence $A^2 - 7A = \begin{pmatrix} 18 & 7 \\ 14 & 11 \end{pmatrix} - \begin{pmatrix} 28 & 7 \\ 14 & 21 \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}$ **1**

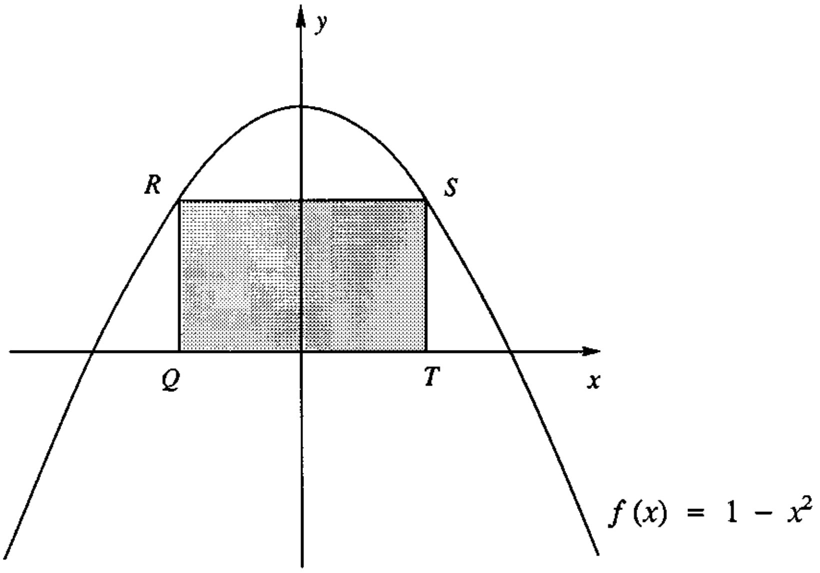
which cI for $c = -10$. **1**

$$A(A - 7I) = -10I$$

$$\Rightarrow B = -\frac{1}{10}(A - 7I)$$ **1**

$$= -\frac{1}{10}\begin{pmatrix} -3 & 1 \\ 2 & -4 \end{pmatrix}$$ **1**

The rectangle $QRST$ is situated beneath the graph of the function $f(x) = 1 - x^2$ as shown in the diagram below.



Let T be the point $(t, 0)$. Find the value of t for which the rectangle has the largest area and find this largest area. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.2	5		1.2.8		1997 SY1 Q7

$$QT = 2t \text{ and } ST = 1 - t^2$$ **1**

$$\text{Area} = A(t) = 2t(1 - t^2)$$ **1**

$$A(t) = 2t - 2t^3$$

$$A'(t) = 2 - 6t^2 = 0 \text{ at SV}$$

$$t = \pm\frac{1}{\sqrt{3}}$$ **1**

$$A''(t) = -12t < 0 \text{ when } t > 0.$$ **1**

$$\text{Maximum value} = A\left(\tfrac{1}{\sqrt{3}}\right) = \tfrac{2}{\sqrt{3}}\left(1 - \tfrac{1}{3}\right) = \tfrac{4}{3\sqrt{3}}.$$ **1**

(a) Find partial fractions for

$$\frac{4}{x^2 - 4}. \qquad 2$$

(b) By using (a) obtain

$$\int \frac{x^2}{x^2 - 4}\, dx. \qquad 4$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	1.1	2		1.1.6		1997 SY1 Q9
	4	2.2		4	2.2.1	1.1.6	

(a)

$$\frac{4}{x^2 - 4} = \frac{A}{x - 2} + \frac{B}{x + 2}$$ **1**

$$4 = A(x + 2) + B(x - 2)$$

$$x = 2 \Rightarrow 4A = 4 \Rightarrow A = 1$$

$$x = -2 \Rightarrow -4B = 4 \Rightarrow B = -1$$ **1 for *A* and *B***

$$\therefore \frac{4}{x^2 - 4} = \frac{1}{x - 2} - \frac{1}{x + 2}$$

(b)

$$\int \frac{x^2}{x^2 - 4}\, dx = \int 1 + \frac{4}{x^2 - 4} dx$$ **1 for division**

$$= \int 1 + \frac{1}{x - 2} - \frac{1}{x + 2} dx$$ **1 for applying (a)**

$$= x + \ln(x - 2) - \ln(x + 2) + c$$ **2, less 1 for each error**

For $f(x) = \frac{1}{3}x^3 + \frac{3}{2}x^2 + x, x \in \mathbf{R}$, find all the values of x for which $|f'(x)| < 1$. **6**
[A solution to this question which relies on a graphical or numerical approach will receive little or no credit.]

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	1.2	6		1.2.8		1997 SY1 Q10

$$f(x) = \tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + x$$ **1**

$$f'(x) = x^2 + 3x + 1$$

$$f'(x) = 1 \text{ or } f'(x) = -1$$ **1**

$$f'(x) = 1 \Rightarrow x^2 + 3x = 0$$

$$\Rightarrow x = 0 \text{ or } x = -3$$ **1**

$$f'(x) = -1 \Rightarrow x^2 + 3x + 2 = 0$$

$$\Rightarrow x = -1 \text{ or } x = -2$$ **1**

$$\therefore -1 < f'(x) < 1$$

$$\text{for } -3 < x < -2$$ **1**

$$\text{and } -1 < x < 0.$$ **1**

(a) Find the modulus and argument of the complex number $2 + 2\sqrt{3}i$ and plot it on an Argand diagram. **3**

(b) By writing $z = r(\cos\theta + i\sin\theta)$, obtain a value for r and values for θ such that

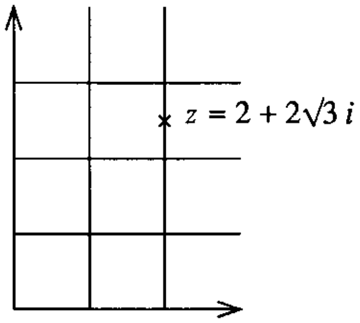
$$z^2 = 2 + 2\sqrt{3}i.$$

Plot the complex numbers z on the same Argand diagram as in part (a). **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.3	3		2.3.7	2.3.4	1997 SY1 Q13
(b)	6	2.3		6	2.3.11	2.3.13	

(a) $z = 2 + 2\sqrt{3}\,i \Rightarrow |z| = \sqrt{4+12} = 4$ **1**

and $\arg z = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = 60°$ (or $\frac{\pi}{3}$). **1**



1

(b)

$$z = r(\cos\theta + i\sin\theta)$$

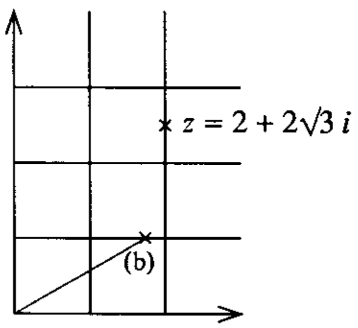
$$\Rightarrow z^2 = r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)$$ **1 for r^2**

$$= r^2(\cos 2\theta + i\sin 2\theta)$$ **1 for 2θ**

$$\therefore r^2 = 4 \Rightarrow \text{ and } 2\theta = \frac{\pi}{3}$$ **1**

$$\Rightarrow r = 2$$ **1**

$$\Rightarrow \theta = \frac{\pi}{6}.$$ **1**



1

An accident at a factory on a river results in the release of a polluting chemical. Immediately after the accident, the concentration of the chemical in the river becomes k g/m^3. The river flows at a constant rate of w m^3 /hour into a loch of volume V m^3. Water flows over the dam at the other end of the loch at the same rate. The level, x g/m^3, of the pollutant in the loch t hours after the accident satisfies the differential equation

$$V\frac{dx}{dt} = w(k - x).$$

(a) Find the general solution for x in term of t. **4**

(b) In this particular case, the values of the constants are $V = 16000000$, $w = 8000$ and $k = 1000$.

(i) Before the accident, the level of chemical in the loch was zero. Fish in the loch will be poisoned if the level of pollutant in the loch reaches 10 g/m^3. How long do the authorities have to stop the leak before this level is reached? **3**

(ii) In fact, when the level of the pollutant in the loch has reached 5 g/m^3, the leak is located and plugged. The level in the river then drops to zero and the level in the loch falls according to the differential equation

$$V\frac{dx}{dt} = -wx.$$

According to European Union standards, 1 g/m^3 is a safe level for the chemical. How much longer will it be before the level in the loch drops to this value? **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.2	4		2.2.5		1997 SY1 Q14
(b)(i)	3	2.2	3		2.2.5		
(b)(ii)	5	2.2		5	2.2.5		

(a)

$$V\frac{dx}{dt} = w(k - x)$$

$$\int \frac{dx}{k - x} = \int \frac{w}{V}dt \qquad \textbf{1}$$

$$-\ln(k - x) = \frac{wt}{V} + c \qquad \textbf{2 (1 for each side)}$$

$$k - x = \exp\left(-\frac{wt}{V} - c\right)$$

$$x = k - \exp\left(-\frac{wt}{V} - c\right) = k - Ae^{-wt/V} \qquad \textbf{1 for making } \boldsymbol{x} \textbf{ subject}$$

(b) $V = 16\,000\,000, w = 8000, k = 1000$

(i) when $t = 0,\ x = 0.$

$$0 = 1000 - A \Rightarrow A = 1000 \qquad \textbf{1 for a constant}$$

$$\text{i.e. } x = 1000\left(1 - e^{-t/2000}\right)$$

$$\text{when } x = 10 \qquad 10 = 1000\left(1 - e^{-t/2000}\right) \qquad \textbf{1 for putting in 10}$$

$$e^{-t/2000} = 1 - 0.01$$

$$t = -2000 \ln 0.99$$

$$= 20.1 \text{ hours} \qquad \textbf{1 for evaluating}$$

(ii)

$$V\frac{dx}{dt} = -wx$$

$$\int \frac{dx}{x} = -\int \frac{w}{V}dt \qquad \textbf{1 for separating}$$

$$\ln x = -\frac{wt}{V} + c'$$

$$x = Be^{-wt/V} \qquad \textbf{1 for general case}$$

$$\text{when } t = 0, x = 5 \Rightarrow B = 5 \qquad \textbf{1}$$

$$\text{i.e. } x = 5e^{-t/2000}$$

$$\text{so when } x = 1,\ e^{-t/2000} = 0.2 \qquad \textbf{1}$$

$$t = -2000 \ln 0.2$$

$$\approx 3220 \text{ hours!} \qquad \textbf{1}$$

Make use of the fact that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

to write down a formula for $\sum_{k=1}^{n} k^3$ in terms of n. **2**

Use this formula to obtain an expression for $\sum_{k=1}^{n}(2k^3 + k^2 - k)$ in terms of n.

Express your answer in fully factorised form. **6**

Hence evaluate

$$(2 + 1 - 1) + (16 + 4 - 2) + \dots + (2000 + 100 - 10).$$ **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	2.5	2		2.5.8		1997 SY1 Q
	6	3.5	6		3.5.8	3.5.6	
	2	3.5		2	3.5.8		

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

$$= (\tfrac{1}{2}n(n+1))^2$$ **1**

$$= \tfrac{1}{4}n^2(n+1)^2$$ **1**

$$\sum_{=1}^{n}(2k^3 + k^2 - k) = \sum_{k=1}^{n} 2k^3 + \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k$$ **1**

$$= 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k$$ **1**

$$= \tfrac{1}{2}n^2(n+1)^2 + \tfrac{1}{6}n(n+1)(2n+1) - \tfrac{1}{2}n(n+1)$$ **1 for Σk^2**

$$= \tfrac{1}{6}n(n+1)[3n^2 + 3n + 2n + 1 - 3]$$

$$= \tfrac{1}{6}n(n+1)(3n^2 + 5n - 2)$$

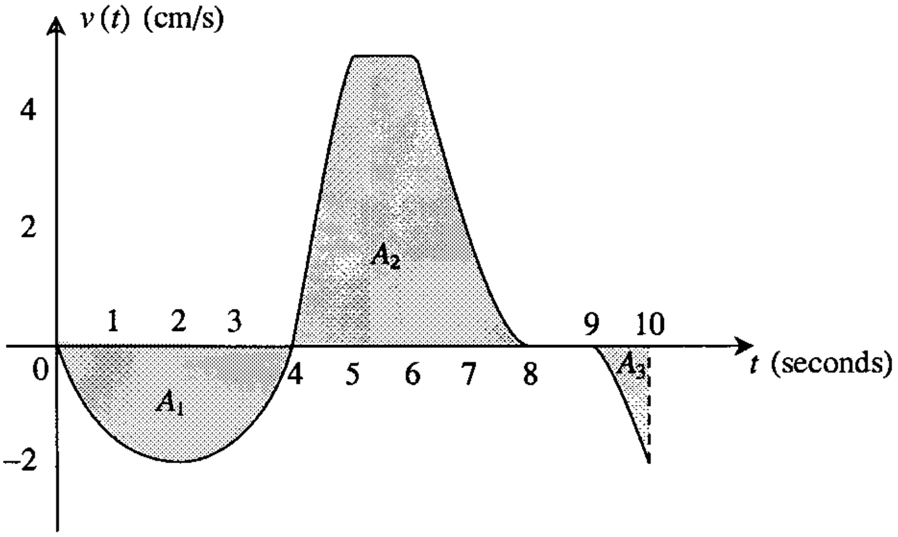
$$= \tfrac{1}{6}n(n+1)(n+2)(3n-1)$$

3 marks for the manipulation, 1 off for each error (down to 0)

$$(2 + 1 - 1) + (16 + 4 - 2) + \dots + (2000 + 100 - 10)$$

$$= \sum_{k=1}^{10}(2k^3 + k^2 - k)$$ **1 for using $n = 10$**

$$= \tfrac{1}{6} \times 10 \times 11 \times 12 \times 29 = 6380$$ **1 for evaluating**



The graph above represents the **velocity** v of the cutting head of a computer controlled milling machine. **Positive** values of v represent movement to the right.

(a) What is represented, in terms of the movement of the cutting head, by the gradient $\frac{dv}{dt}$ of this graph at a time t? **1**

(b) What is represented by the integral $\int_0^T v(t)\,dt$ for some time T? **2**

The areas marked on the graph have values:

$$A_1 = 6, \qquad A_2 = 13, \qquad A_3 = 1.$$

(c) What is the maximum displacement of the cutting head from its starting position and what is the smallest value of t for which this maximum is attained? **2**

(d) What is the maximum acceleration of the cutting head? When this is attained, is the head moving to the right or to the left? **2**

(e) What is the total distance moved by the cutting head for $0 \leqslant t \leqslant 10$? **2**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	1	1.4	1		1.4.1		1997 SY1 Q16
(b)	2	1.4	2				
(c)	2	1.4	2		1.4.1		
(d)	2	1.4		2			
(e)	2	1.4		2			

(a) The gradient represents the **acceleration** of the cutting head. **1**

(b) The integral $\int_0^T v(t)\,dt$ represents the **distance** the head is **1**
from its **starting point.** **1**

(c) $A_1 = -6; A_1 + A_2 = 13 - 6 = 7;$ **1 for the idea of adding up**
Total shaded area is $13 - 6 - 1 = 6$.
The maximum displacement is 7 cm and occurs at time $t = 8$. **1**

(d) The maximum acceleration is when the graph is steepest which is between $t = 4$ and $t = 5$.
The speed increases by 5 cm/s during this time
so the acceleration is **5 cm/s²**. **1**
At this time the head is moving to the **right.** **1**

(e) The total distance is the **sum of the magnitudes of** A_1, A_2, A_3 **1**
i.e. 20 cm. **1**

A sequence $\{a_n\}$ is defined by $a_1 = 4$, $a_n = 3a_{n-1} - 5$ $(n \geqslant 2)$. Prove by induction that

$$a_n = \frac{1}{2}(3^n + 5)$$

for all positive integers n. **5**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.5	5		3.5.6		1997 SY2 Q1

True for $n = 1$ since $\frac{1}{2}(3 + 5) = 4 (= a_1)$. **1**

Suppose

$$a_k = \frac{1}{2}(3^k + 5)$$ **1**

then

$$a_{k+1} = 3 \times \frac{1}{2}(3^k + 5) - 5$$ **1**

$$= \frac{1}{2} \times 3^{k+1} + \frac{15}{2} - 5 = \frac{1}{2} \times 3^{k+1} + \frac{5}{2}$$

$$= \frac{1}{2}(3^{k+1} + 5)$$ **1**

So the result is true for $n = 1$, and is true for $n = k + 1$ whenever it is true for $n = k$. So it is true for all $n \geqslant 1$. **1**

Use the Euclidean Algorithm to find integers x, y such that

$$29x + 113y = 1.$$ **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	2	3.5	2		3.5.10		1997 SY2 Q2
	2	3.5		2	3.5.11		

$$113 = 3 \times 29 + 26$$ **1**

$$29 = 1 \times 26 + 3$$

$$26 = 8 \times 3 + 2$$

$$3 = 1 \times 2 + 1$$ **1**

So

$$1 = 3 - 1 \times (26 - 8 \times 3) = 9 \times 3 - 1 \times 26$$ **1**

$$= 9(29 - 26) - 26 = 9 \times 29 - 10 \times 26$$

$$= 9 \times 29 - 10(113 - 3 \times 29)$$

$$= 39 \times 29 - 10 \times 113$$ **1**

So $x = 39$, $y = -10$ satisfy request.

The lines L_1 and L_2 have equations

$$L_1 : \frac{x-2}{2} = \frac{y+5}{3} = \frac{z-5}{-1};$$

$$L_2 : \frac{x+6}{4} = \frac{y+2}{1} = \frac{z+4}{2}.$$

Determine whether or not L_1 and L_2 intersect. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	3.1	6		3.1.9		1997 SY2 Q3

$$L_1 : x = 2s + 2, y = 3s - 5, z = -s + 5$$ **1**

$$L_2 : x = 4t - 6, y = t - 2, z = 2t - 4$$ **1**

Equating the x and the y coordinates gives

$$2s + 2 = 4t - 6, \qquad 3s - 5 = t - 2$$ **1**

Solving these

$$\begin{array}{c} s - 2t = -4 \\ 3s - t = 3 \end{array} \Rightarrow \begin{array}{c} 3s - 6t = -12 \\ 3s - t = 3 \end{array} \Rightarrow 5t = 15$$

$$\Rightarrow t = 3, s = 2$$ **1**

Substituting back gives $z = 3$ for L_1 and $z = 2$ for L_2 **1**

so the lines do **not** intersect. **1**

It is given that $T_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$, where $\alpha \in \mathbf{R}$.

(i) Evaluate $\det T_\alpha$.

(ii) Show that $T_\alpha T_\beta$ is a rotation matrix.

(iii) Express $T_\alpha T_\beta T_\alpha$ as T_γ for some γ. **6**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	1	3.2	1		3.2.4		1997 SY2 Q5
(ii)	3	3.2	3		3.2.9		
(iii)		3.2			3.2.		

(i) $$\det T_\alpha = -\cos^2\alpha - \sin^2\alpha \qquad \mathbf{1}$$

(ii) $$T_\alpha T_\beta = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha\cos\beta + \sin\alpha\sin\beta & \cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta - \cos\alpha\sin\beta & \sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} \qquad \mathbf{1}$$

$$= \begin{pmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix} \qquad \mathbf{1}$$

$$= R_{\alpha-\beta} \qquad \mathbf{1}$$

(iii) $$T_\alpha T_\beta T_\alpha = \begin{pmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix} \qquad \mathbf{1}$$

$$= \begin{pmatrix} \cos(2\alpha-\beta) & \sin(2\alpha-\beta) \\ \sin(2\alpha-\beta) & -\cos(2\alpha-\beta) \end{pmatrix} \qquad \mathbf{1}$$

$$= T_{2\alpha-\beta}$$

(a) Let $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$ and $N = \begin{pmatrix} c & d \\ d & c \end{pmatrix}$, where a, b, c and $d \in \mathbf{R}$.

Find MN and, by considering determinants, derive the identity

$$(a^2 - b^2)(c^2 - d^2) = (ac + bd)^2 - (ad + bc)^2. \qquad \mathbf{3}$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	3.2	3		3.2.7		1997 SY2 Q9

(a)

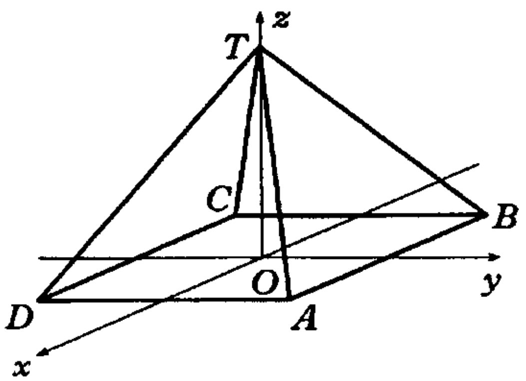
$$MN = \begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} c & d \\ d & c \end{pmatrix} = \begin{pmatrix} ac + bd & ad + bc \\ bc + ad & bd + ac \end{pmatrix} \qquad \mathbf{1}$$

Since $\det M \det N = \det MN$ **1**

we have

$$(a^2 - b^2)(c^2 - d^2) = (ac + bd)^2 - (ad + bc)^2 \qquad \mathbf{1}$$

The points $A(1, 1, 0)$, $B(-1, 1, 0)$, $C(-1, -1, 0)$, and $D(1, -1, 0)$ are the corners of the square base of a pyramid whose four sloping faces are equilateral triangles. The apex, T, of the pyramid is the point $(0, 0, t)$, where $t > 0$.



(a) Show that $t = \sqrt{2}$. **2**

(b) Obtain an equation of the plane containing the triangle ADT. **4**

(c) The angle between two faces is defined to be the acute angle between their normals.

Find:

(i) the angle between two adjacent triangular faces;

(ii) the angle between a triangular face and the base of the pyramid. **5**

(d) The line L through the origin perpendicular to the face ADT meets ADT at G. Find an equation for L and hence, or otherwise, obtain the coordinates of G. **4**

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2		2				1997 SY2 Q10
(b)	4	3.1	4		3.1.7		
(c)(i)	2	3.1		2	3.1.8		
(c)(ii)	3	3.1		3	3.1.8		
(d)	4	3.1		4	3.1.9		

(a) $$|AD| = 2, \qquad |AT| = \sqrt{1 + 1 + t^2}$$ **1**

so require $4 = 2 + t^2$, i..e. $t^2 = 2$ and so $t = \sqrt{2}$. **1**

(b) A normal to the plane is

$$\overrightarrow{DA} \times \overrightarrow{DT}$$ **1**

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -1 & 1 & \sqrt{2} \end{vmatrix} = 2\sqrt{2}\,\mathbf{i} + 2\,\mathbf{k}$$ **1**

So an equation of the plane is

$$\sqrt{2}x + z = k \qquad \text{(for some } k\text{)}$$ **1**

Since $(0,\ 0,\ \sqrt{2})$ lies on plane, $k = \sqrt{2}$, so equation is

$$\sqrt{2}x + z = \sqrt{2}$$ **1**

(c) (i) A normal to the plane ABT is

$$\overrightarrow{AB} \times \overrightarrow{AT} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 0 & 0 \\ -1 & -1 & \sqrt{2} \end{vmatrix} = 2\sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}$$ **1**

so the angle, θ, between the planes is given from

$$\sqrt{3} \times \sqrt{3}\cos\theta = \left(2\sqrt{2}\,\mathbf{i} + 2\,\mathbf{k}\right).\left(2\sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}\right) = 1$$ **1**

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{3}\right)$$ **1**

(ii) Consider the angle between ADT and base. Normals are $2\sqrt{2}\,\mathbf{i} + 2\,\mathbf{k}$ (or $\sqrt{2}\,\mathbf{i} + \mathbf{k}$) and $\mathbf{k}$, so the angle ϕ satisfies **1**

$$\sqrt{3} \times 1\cos\phi = \left(\sqrt{2}\,\mathbf{i} + \mathbf{k}\right).\mathbf{k} = 1$$

$$\Rightarrow \qquad \phi = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$ **1**

(d) Equation of L is

$$\frac{x-0}{\sqrt{2}} = \frac{y-0}{0} = \frac{z-0}{1} (= s)$$ **1**

so a general point on L is

$$x = \sqrt{2}s, \qquad y = 0, \qquad z = s.$$ **1**

This lies on the face ADT, which has equation $\sqrt{2}x + z = \sqrt{2}$, if

$$2s + s = \sqrt{2}$$ **1**

$$\text{i.e. } s = \frac{\sqrt{2}}{3} \text{ so } G \text{ is the point } \left(\frac{2}{3},\ 0,\ \frac{\sqrt{2}}{3}\right)$$ **1**

Differentiate

$$g(x) = \frac{\sin x}{1 + \cos x}, \qquad -\pi < x < \pi,$$

and simplify your answer. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	1.2	3		1.2.4		1998 SY1 Q1

$$g(x) = \frac{\sin x}{1 + \cos x}$$

$$g'(x) = \frac{\cos x(1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}$$ **1 method (quotient)**

1 for accuracy

$$= \frac{\cos x + 1}{(1 + \cos x)^2}$$ **1 for simplifying**

$$= \frac{1}{1 + \cos x}$$

Use the substitution $u = t + 1$ to find

$$\int_0^8 \frac{t + 2}{\sqrt{(t + 1)}}\, dt. \qquad (5)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1·3	5		1·3·4		1998 SY 1 Q3

$$u = t + 1 \text{ so } \begin{cases} t = 8 \Rightarrow u = 9 \\ t = 0 \Rightarrow u = 1 \end{cases} \qquad \mathbf{1}$$

$$du = dt$$

$$\int_0^8 \frac{t + 2}{\sqrt{(t + 1)}}\, dt = \int_1^9 \frac{u + 1}{\sqrt{u}}\, du \qquad \mathbf{1}$$

$$= \int_1^9 \{u^{1/2} + u^{-1/2}\}\, du \qquad \mathbf{1}$$

$$= \left[\tfrac{2}{3}u^{\frac{3}{2}} + 2u^{\frac{1}{2}}\right]_1^9 \qquad \mathbf{1}$$

$$= [\tfrac{2}{3} \times 27 + 2 \times 3] - [2\tfrac{2}{3}] \qquad \mathbf{1}$$

$$= 21\tfrac{1}{3}.$$

Use calculus to find all the values of x for which the function

$$f(x) = (1+x)^2 e^{-x}, \qquad x \in \mathbf{R},$$

is increasing. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.2		5	1.2.8		1998 SY 1 Q6

$$f(x) = (1+x)^2 e^{-x}$$

$$f'(x) = 2(1+x)e^{-x} - (1+x)^2 e^{-x}$$ **1 method (product)**

1 for accuracy

> 0 for $f(x)$ to be increasing. **1**

$$(1+x)\left[2-(1+x)\right]e^{-x} > 0$$

$$(1+x)(1-x)e^{-x} > 0$$ **1 for factorising**

$x^2 < 1 \quad \Rightarrow -1 < x < 1$ **1**

($-1 \leqslant x \leqslant 1$ is acceptable)

Verify that $z = 1 + i$ is a solution of the equation

$$z^3 + 16z^2 - 34z + 36 = 0. \quad (3)$$

Write down a second solution of the equation. (1)

Hence find constants α and β such that

$$z^3 + 16z^2 - 34z + 36 = (z^2 - \alpha z + \alpha)(z + \beta). \quad (3)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	7	2.3	4	3	2.3.9		1998 SY1 Q7

$(1 + i)^3 + 16(1 + i)^2 - 34(1 + i) + 36$

$= (-2 + 2i) + 16(2i) - 34 - 34i + 36$ **1 for $(1 + i)^2$**
1 for $(1 + i)^3$

$= 0 + 0i$ **1 for rest**

$(1 - i)$ is also a root. **1**

Factor is

$[z - (1 + i)][z - (1 - i)] = [(z - 1) - i][(z - 1) + i]$ **1 for method**

$= (z - 1)^2 + 1 = z^2 - 2z + 2.$ **1**

So $z^3 + 16z^2 - 34z + 36 = (z^2 - 2z + 2)(z + 18)$

i.e. $\alpha = 2, \beta = 18.$ **1 for values**

(trial and error is not valid)

Show that

$$\int_0^t x \sin(\pi x)\, dx = \frac{1}{\pi^2}\sin(\pi t) - \frac{t}{\pi}\cos(\pi t). \qquad (4)$$

Use this result to find the **exact** value of $\int_0^{½} x \sin(\pi x)\, dx$. (1)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	2.2	5		2.2.2		1998 SY1 Q8

$\int_0^t x \sin(\pi x)\, dx$

$$= \left[x \int \sin(\pi x)\, dx\right]_0^t - \left[\int \int \sin \pi x\, dx.dx\right]_0^t$$ **1 method (parts)**

$$= \left[-\frac{x}{\pi}\cos(\pi x)\right]_0^t - \left[-\frac{1}{\pi}\int \cos(\pi x)\, dx\right]_0^t$$ **1 for $\int \sin \pi x\, dx$**

$$= \left[-\frac{x}{\pi}\cos(\pi x)\right]_0^t - \left[-\frac{1}{\pi^2}\sin(\pi x)\right]_0^t$$ **1 for $\int \cos \pi x\, dx$**

$$= -\frac{t}{\pi}\cos \pi t + \frac{1}{\pi^2}\sin \pi t$$

$$= \frac{1}{\pi^2}\sin(\pi t) - \frac{t}{\pi}\cos(\pi t).$$ **1 for final step – evidence needed**

$$\int_0^{1/2} x \sin(\pi x)\, dx = \frac{1}{\pi^2}$$ **1 for finishing**

Express $\dfrac{1}{(2x-1)(2x+1)}$ in partial fractions. (2)

Deduce that

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right). \qquad (2)$$

Evaluate

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)(2k+1)}. \qquad (1)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.4	2	3	2.4.7	1.1.6	1998 SY1 Q9

$$\frac{1}{(2x-1)(2x+1)} = \frac{A}{2x-1} + \frac{B}{2x+1}$$ **1 method**

$$1 = A(2x+1) + B(2x-1)$$

$$\Rightarrow A = \tfrac{1}{2},\ B = -\tfrac{1}{2}$$ **1**

Thus $\dfrac{1}{(2x-1)(2x+1)} = \dfrac{1}{2(2x-1)} - \dfrac{1}{2(2x+1)}$

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)}$$

$$= \left[\frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}\right] + \left[\frac{1}{2(2n-3)} - \frac{1}{2(2n-1)}\right] + \dots + \left[\frac{1}{2} - \frac{1}{6}\right]$$ **1 for listing**

$$= \frac{1}{2} - \frac{1}{2(2n+1)}$$ **1 for cancelling**

$$= \frac{1}{2}\left[1 - \frac{1}{2n+1}\right]$$

$$\to \frac{1}{2} \text{ as } n \to \infty.$$ **1**

A car is travelling along a straight road at a constant velocity of 19 ms^{-1}. Spotting that the road is blocked some distance ahead by a fallen tree, the driver brakes immediately and brings the car to a complete stop T seconds later. The car's acceleration t seconds after the brakes are applied is $a(t)$ ms^{-2}, where

$$a(t) = -\tfrac{3}{2}\sqrt{4 + t}, \qquad 0 \leqslant t \leqslant T.$$

(a) Show that $T = 5$. (4)

(b) Calculate the distance travelled by the car during the 5 seconds that it takes to come to a stop. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.2	4		1.2.8		1998 SY1 Q10
(b)	3	1.2	3		1.2.8		

(a) $a(t) = -\tfrac{3}{2}\sqrt{4 + t}$ so

$$v(t) = \int a(t)\,dt = \int -\frac{3}{2}(4 + t)^{\frac{1}{2}}\,dt$$ **1**

$$= -(4 + t)^{\frac{3}{2}} + c$$ **1**

Since $v(0) = 19, c = 27$ **1**

and hence

$$v(t) = 27 - (4 + t)^{\frac{3}{2}}.$$

Stops when $v(t) = 0$ i.e. $(4 + T)^{\frac{3}{2}} = 27 \Rightarrow 4 + T = 9 \Rightarrow T = 5.$ **1**

(b)

$$s(t) = \int v(t)\,dt = 27t - \frac{2}{5}(4 + t)^{\frac{5}{2}} + c'.$$ **1**

$$s(0) = 0 \Rightarrow c' = \frac{64}{5}$$ **1**

$$s(5) = 135 - \tfrac{2}{5}(9)^{\frac{5}{2}} + \tfrac{64}{5}$$

$$= 135 - \frac{422}{5} = 50.6.$$ **1**

In a town with population 40 000, a 'flu virus spread rapidly last winter. The **percentage** P of the population infected t days after the initial outbreak satisfies the differential equation

$$\frac{dP}{dt} = kP, \qquad \text{where } k \text{ is a constant.}$$

(a) If 100 people are infected initially, find, in terms of k, the percentage infected t days later. (4)

(b) Given that 500 people have 'flu after 7 days, how many more are likely to have contracted the virus after 10 days? (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.2	4		2.2.5		1998 SY1 Q12
(b)	3	2.2		3	2.2.5		1998 SY1 Q12

(a)

$$\int \frac{dP}{P} = \int k\,dt \qquad \textbf{1}$$

$$\ln P = kt + c \qquad \textbf{1}$$

$$t = 0,\ P = 0.25 \Rightarrow c = \ln 0.25 \qquad \textbf{1}$$

$$\ln(P \div 0.25) = kt$$

$$4P = e^{kt} \Rightarrow P = \tfrac{1}{4}e^{kt} \qquad \textbf{1 for a formula}$$

(b) $t = 7,\ P = 1.25$

$$e^{7k} = 5 \Rightarrow e^{k} = 5^{\frac{1}{7}} \qquad \textbf{1 for } k \textbf{ (alternatives possible)}$$

$$P = \tfrac{1}{4}\left(5^{\frac{1}{7}}\right)^{10} \approx 2.49\ldots \qquad \textbf{1}$$

$$2.49\% \text{ of } 40000 = 996.6\ldots$$

This would be an increase of about 497. **1**

Let the function f be given by

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x - 2)^2}, \qquad x \neq 2.$$

(a) The graph of $y = f(x)$ crosses the y-axis at $(0, a)$. State the value of a. (1)

(b) For the graph of $f(x)$

(i) write down the equation of the vertical asymptote, (1)

(ii) show algebraically that there is a non-vertical asymptote and state its equation. (3)

(c) Find the coordinates and nature of the stationary point of $f(x)$. (4)

(d) Show that $f(x) = 0$ has a root in the interval $-2 < x < 0$. (1)

(e) Sketch the graph of $y = f(x)$. (You must include on your sketch the results obtained in the first four parts of this question.) (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	1	1.4	1		1.4.9		1998 SY1 Q13
(b)	4	1.4	4		1.4.9		
(c)	4	1.4	4		1.4.9		
(d)	1	1.4	1		1.4.9		
(e)	2	1.4	2		1.4.9		

$$f(x) = \frac{2x^3 - 7x^2 + 4x + 5}{(x-2)^2}$$

(a) $x = 0 \Rightarrow y = \frac{5}{4}$ $a = \frac{5}{4}$ **1**

(b) (i) $x = 2$ **1**

(ii) After division, the function can be expressed in quotient/remainder form:–

$$f(x) = 2x + 1 + \frac{1}{(x-2)^2}$$

1 for method
1 for accuracy

Thus the line $y = 2x + 1$ is a slant asymptote. **1**

(c) From (b), $f'(x) = 2 - \frac{2}{(x-2)^3}$. **1 for any correct derivative**

Turning point when

$$2 - \frac{2}{(x-2)^3} = 0$$

$$(x-2)^3 = 1$$

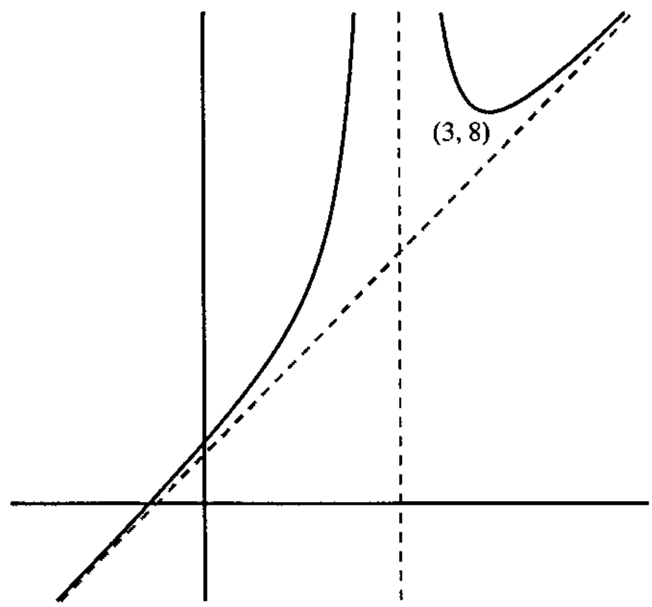
$$x - 2 = 1 \Rightarrow x = 3$$ **1**

$$f''(x) = \frac{6}{(x-2)^4} > 0 \text{ for all } x.$$ **1**

The stationary value at (3, 8) is a minimum turning point. **1 for y value**

(d) $f(-2) = \frac{-16-28-8+5}{(-4)^2} < 0$;
$f(0) = \frac{5}{4} > 0$. **1 for showing change of sign**
Hence a root between −2 and 0.

(e)



1 for general shape
1 for *showing* asymptotes

The matrices A and B are defined by

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & 9 \\ 4 & -8 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 71 & a & -6 \\ 34 & b & -3 \\ -12 & c & 1 \end{pmatrix}$$

where a, b and c are constants.

(a) Find the matrix $B - 3A$. (2)

(b) (i) Verify that $AB = I$, where I is the 3×3 identity matrix, provided that

$$a - b + 3c = 0$$

$$2a - b + 9c = 1$$

$$4a - 8b + c = 0. \qquad (3)$$

(ii) Use Gaussian elimination to find the values of a, b and c for which $AB = I$. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	1.5	2		1.5.1		1998 SY1 Q14
(b)	7	1.5	7		1.5.4		

$$A = \begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & 9 \\ 4 & -8 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 71 & a & -6 \\ 34 & b & -3 \\ -12 & c & 1 \end{pmatrix}$$

(a)

$$B - 3A = \begin{pmatrix} 71 & a & -6 \\ 34 & b & -3 \\ -12 & c & 1 \end{pmatrix} - \begin{pmatrix} 3 & -3 & 9 \\ 6 & -3 & 27 \\ 12 & -24 & 3 \end{pmatrix}$$ **1 for matrix 3A**

$$= \begin{pmatrix} 68 & a+3 & -15 \\ 28 & b+3 & -30 \\ -24 & c+24 & -2 \end{pmatrix}$$ **1**

(b) (i)

$$\begin{pmatrix} 1 & -1 & 3 \\ 2 & -1 & 9 \\ 4 & -8 & 1 \end{pmatrix} . \begin{pmatrix} 71 & a & -6 \\ 34 & b & -3 \\ -12 & c & 1 \end{pmatrix}$$ **1**

$$= \begin{pmatrix} 1 & a-b+3c & 0 \\ 0 & 2a-b+9c & 0 \\ 0 & 4a-8b+c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **1**

Provided

$$a - b + 3c = 0$$

$$2a - b + 9c = 1$$ **1**

$$4a - 8b + c = 0$$

(ii)

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & -1 & 9 & 1 \\ 4 & -8 & 1 & 0 \end{array}\right)$$ **1 for method**

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & -4 & -11 & 0 \end{array}\right) \quad \begin{array}{l} r_2' = r_2 - 2r_1 \\ r_3' = r_3 - 4r_1 \end{array}$$ **1**

$$\left(\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right) \quad r_3'' = r_3' + 4r_2'$$ **1**

$$c = 4$$

$$b = -11$$

$$a = -23$$ **1**

(a) Find the derivative of the function $g(x) = \sqrt{1 - x^2}$ and hence, or otherwise, obtain the indefinite integral

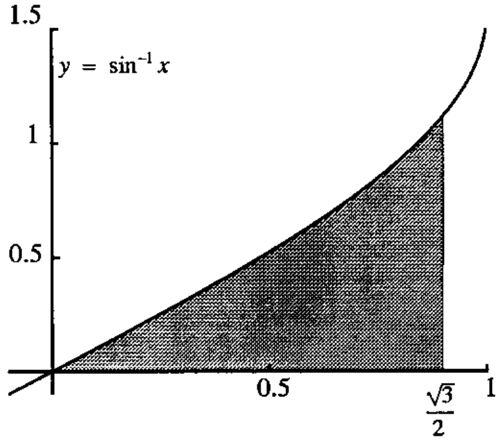
$$\int \frac{x}{\sqrt{1 - x^2}}\, dx.$$ (3)

(b) Use integration by parts to show that

$$\int f(x)\, dx = xf(x) - \int xf'(x)\, dx$$

where f is a differentiable function with derivative f'. (2)

(c) The diagram below represents the graph of $y = \sin^{-1} x$. Use the previous results to calculate the area of the shaded region which lies between $x = 0$ and $x = \frac{\sqrt{3}}{2}$. (4)



			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.2	3		1.2.4	1.3.4	1998 SY1 Q15
(b)	2	2.2		2	2.2.2		1998 SY1 Q15
(c)	4	1.3		4	1.3.6		1998 SY1 Q15

(a) $g(x) = (1 - x^2)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}(-2x)(1 - x^2)^{-\frac{1}{2}} = \frac{-x}{\sqrt{1 - x^2}}$ **1 for using chain rule**

1 for accuracy

$\therefore \int \frac{x}{\sqrt{1 - x^2}}\,dx = -\sqrt{1 - x^2} + c$ **1 for integral**

(*c* not essential)

(b) $\int 1.f(x)\,dx = f(x)\int 1.dx - \int f'(x).x\,dx$ **1 for introducing '1'**

$= x\,f(x) - \int x f'(x)\,dx$ **1 for accuracy**

(c) $\int_0^{\frac{\sqrt{3}}{2}} \sin^{-1} x\,dx = \left[x \sin^{-1} x\right]_0^{\frac{\sqrt{3}}{2}} - \int_0^{\frac{\sqrt{3}}{2}} \frac{x}{\sqrt{1 - x^2}}\,dx$ **1 for using result from (b)**

1 for derivative of $\sin^{-1} x$

$= \left[\frac{\sqrt{3}}{2}\frac{\pi}{3}\right] + \left[\sqrt{1 - x^2}\right]_0^{\frac{\sqrt{3}}{2}}$ **1 for using part (a)**

$= \frac{\pi\sqrt{3}}{6} + \frac{1}{2} - 1$ **1 for replacing limits**

$= \frac{1}{6}(\pi\sqrt{3} - 3)$

(a) A toy manufacturer is planning to launch a new product on the market. The Sales Manager believes that the recurrence relation

$$u_n = u_{n-1} + 5000 \qquad n = 2, 3, 4, \ldots$$

provides a reasonable estimate, £u_n, of the profit that will be made from sales of this product during the nth month after it is launched.

Given that $u_1 = 20\,000$, obtain simple expressions for u_n and $\sum_{k=1}^{n} u_k$ in terms of n.

Calculate the estimated profit that will be made from sales during the first year following the launch of the product. (4)

(b) The Managing Director objects to the Sales Manager's formula and she insists that the recurrence relation

$$v_n = 0.9v_{n-1} + 5000 \qquad n = 2, 3, 4, \ldots$$

should be used to produce a more cautious forecast of the monthly profits. Express v_2, v_3 and v_4 in terms of v_1. (2)

By recognising a connection with a geometric series, deduce that if $v_1 = 20\,000$ then

$$v_n = 50\,000 - 30\,000\,(0.9)^{n-1} \qquad n = 1, 2, 3, \ldots .$$

Show that $v_n < u_7$ for all n. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.4	4		2.4.2		1998 SY1 Q16
(b)	6	2.4	2	4	2.4.3		1998 SY1 Q16

(a)

$$u_1 = 20000;\ u_2 = 25000;\ u_3 = 30000$$

$$u_n = 20000 + 5000(n-1)$$

$$= 5000(n+3)$$ **1 for u_n**

$$\sum_{k=1}^{n} u_k = \tfrac{n}{2}[40000 + 5000(n-1)]$$ **1 for method**

$$= 2500n(n+7)$$ **1 for result**

When $n = 12$, this formula would give an estimate of the total profits of £570000. **1**

(b)

$$v_n = 0.9v_{n-1} + 5000$$

$$v_2 = 5000 + 0.9v_1$$

$$v_3 = 5000(1 + 0.9) + 0.9^2v_1$$ **1 for v_2 and v_3**

$$v_4 = 5000(1 + 0.9 + 0.9^2) + 0.9^3v_1$$ **1 for v_4**

$$v_n = 5000(1 + 0.9 + \ldots + 0.9^{n-2}) + 0.9^{n-1}v_1$$ **1 for expression**

$$= 5000\,\frac{1-0.9^{n-1}}{1-0.9} + 0.9^{n-1}v_1$$ **1 for sum of GP**

$$= 50000(1-0.9^{n-1}) + 20000\,.\,0.9^{n-1}$$

$$= 50000 - 30000\,.\,0.9^{n-1}$$ **1 for rearranging**

As n gets bigger, v_n increases and approaches a limit of 50000 from below.
From part (a), $u_7 = 5000(7 + 3) = 50\,000$.
Thus $v_n < u_7$ for all n. **1 for showing**

Let

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

Use induction to prove that, for all positive integers n,

$$A^n = \begin{pmatrix} 1 & 0 \\ 1 - 2^n & 2^n \end{pmatrix}.$$

Determine whether or not this formula for A^n is also valid when $n = -1$. (6)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	3.5	6		3.5.6	3.2.2	1998 SY2 Q1

True for $n = 1$, since $\begin{pmatrix} 1 & 0 \\ 1 - 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$. **1**

Suppose true for $n = k \geqslant 1$. Then

$$A^k = \begin{pmatrix} 1 & 0 \\ 1 - 2^k & 2^k \end{pmatrix}$$ **1 for the inductive hypothesis**

Then

$$A^{k+1} = A.A^k = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 - 2^k & 2^k \end{pmatrix}$$ **1 for this statement**

$$= \begin{pmatrix} 1 & 0 \\ -1 + 2 - 2^{k+1} & 2^{k+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 - 2^{k+1} & 2^{k+1} \end{pmatrix}$$

So the result is true for $n = 1$, and is true for $n = k + 1$ whenever it is true for $n = k$. So it is true for all $n \geqslant 1$. **1**

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$ **1**

$$= \begin{pmatrix} 1 & 0 \\ 1 - 2^{-1} & 2^{-1} \end{pmatrix},$$ **1**

so the formula does hold for $n = -1$.

Let A, B, C be the points $(2, 1, 0)$, $(3, 3, -1)$, $(5, 0, 2)$ respectively.
Find $\overrightarrow{AB} \times \overrightarrow{AC}$.
Hence or otherwise obtain an equation for the plane containing A, B and C. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.1	5		3.1.7	3.1.4	SY 2 1998 Q2

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$ **1**

$$= 3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$ **2**

Equation of the plane is of the form

$$3x - 5y - 7z = k.$$ **1**

But A $(2, 1, 0)$ lies in the plane, so

$$6 - 5 - 0 = k.$$ **1 for value of k**

Equation of plane is $3x - 5y - 7z = 1$.

Show that

$$\det\begin{pmatrix} 2 & 2k & 1 \\ 1 & k-1 & 1 \\ 2 & 1 & k+1 \end{pmatrix}$$

has the same value for all values of k. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	3.2	3		3.2.4		SY2 1998 Q4

$$\det\begin{pmatrix} 2 & 2k & 1 \\ 1 & k-1 & 1 \\ 2 & 1 & k+1 \end{pmatrix}$$

$= 2(k^2 - 1 - 1) - 2k(k + 1 - 2) + 1(1 - 2k + 2)$ **1 for strategy** **1 for accuracy**

$= -1$ **1**

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

(i) Determine whether or not $AB = BA$. (1)

(ii) Verify that $A^2 + 3B^2 = 12I$, where I is the 3×3 identity matrix. (1)

(iii) Find AB, AB^2 and AB^3 as multiples of A, and make a conjecture about a general result for AB^n.
Use induction to prove your conjecture. (6)

(iv) It is given that B is invertible, with inverse of the form

$$B^{-1} = \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix}.$$

Write down a system of linear equations which x, y and z must satisfy, and hence find the values of x, y and z. (4)

(v) Verify that $B^2 - B$ is a multiple of I, and hence find B^{-1} in the form $cB + dI$ where c, d are real numbers. Hence check your answer to (iv). (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	1	3.2	1		3.2.2		SY2 1998 Q7
(ii)	1	3.2	1		3.2.2		
(iii)	6	3.5	2	4	3.5.6	3.2.2	
(iv)	4	1.5	4		1.5.4		
(v)	3	3.2		3	3.2.2		

(a)
$$AB = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} = BA \qquad \mathbf{1}$$

(b)
$$A^2 + 3B^2 = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} + 3\begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} = 12I \qquad \mathbf{1}$$

(c)
$$AB = -A, AB^2 = A, AB^3 = -A \qquad \mathbf{1}$$

so conjecture is

$$AB^n = (-1)^n A \qquad \mathbf{1}$$

Conjecture is true for $n = 1$ since

$$AB^1 = AB = -A = (-1)^1 A. \qquad \mathbf{1}$$

Suppose true for $n = k$: $AB^k = (-1)^{k+1}A$ **1**

Then

$$AB^{k+1} = AB^k.B = (-1)^k A.B \qquad \mathbf{1}$$
$$= (-1)^k.(-1)A = (-1)^{k+1}A$$

so result is true for $n = k + 1$. (So result follows by induction.)

(d) Require

$$x - y - z = 1$$
$$-x + y - z = 0$$
$$-x - y + z = 0 \qquad \mathbf{1}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & -2 & 0 & 1 \end{array}\right) \qquad \mathbf{1}\ \mathbf{1}$$

which leads to $y = z = -\frac{1}{2}$ and $x = 0$. **1**

(e)
$$B^2 - B = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = 2I \qquad \mathbf{1}$$

$$\text{So } B(B - I) = 2I$$

$$\text{thus } B^{-1} = \tfrac{1}{2}(B - I) \qquad \mathbf{1}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} \qquad \mathbf{1}$$

(i) Use the Euclidean Algorithm to find integers x, y such that

$$195x + 239y = 1.$$

Hence write down positive integers a, b such that

$$\det\begin{pmatrix} 195 & 239 \\ a & b \end{pmatrix} = 1. \qquad (6)$$

(ii) Let u, v, w be positive integers. For each of the following, decide whether the statement is true or false. Where false, give a counter-example; where true, give a proof.

(a) If u and v both divide w then $u + v$ divides w.

(b) If u divides both v and w then u divides $v + w$.

(c) If u divides v and v divides w then u divides $v + w$. (7)

Write down the converse of statement (b), and determine whether or not this converse is true. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	6	3.5	2	4	3.5.11	3.5.10	SY 2 1998 Q9
(ii)	7	2.5	7		2.5.5	3.5.3	
	2	3.5	2		3.5.1		

(a)

$$239 = 1 \times 195 + 44 \qquad \mathbf{1}$$

$$195 = 4 \times 44 + 19$$

$$44 = 2 \times 19 + 6$$

$$19 = 3 \times 6 + 1 \qquad \mathbf{1}$$

Thus

$$1 = 19 - 3(44 - 2 \times 19) \qquad \mathbf{1}$$

$$= 7(195 - 4 \times 44) - 3 \times 44$$

$$= 7 \times 195 - 31(239 - 195)$$

$$= 38 \times 195 - 31 \times 239 \qquad \mathbf{1}$$

so take $x = 38, y = -31.$ **1**

Thus $a = 38, b = 31.$ **1**

(b) (i) is false **1**

e.g. $u = 3, v = 4, w = 12$ **1**

(ii) is true. **1**

Suppose $u \mid v$ and $u \mid w$. Then $v = ku$, $w = lu$ for some integers k, l. So **1**

$$v + w = (k + l)u \qquad \mathbf{1}$$

i.e. $u \mid (v + w)$.

(iii) is true.

If $u \mid v$ and $v \mid w$. Then $v = ku, w = lv$ for some integers k, l. **1**

So

$$v + w = (k + kl)u. \qquad \mathbf{1}$$

Converse of (b) is

If u divides $v + w$ then u divides both v and w. **1**

False: e.g. $u = 2, v = w = 3.$ **1**

Use Gaussian elimination to solve the system of linear equations

$$x + y + z = 0$$
$$2x - y + z = -1.1$$
$$x + 3y + 2z = 0.9. \qquad (5)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	1.5	5		1.5.4	1.5.5	1999 SY1 Q1

$$x + y + z = 0$$
$$2x - y + z = -1.1$$
$$x + 3y + 2z = 0.9$$

$x + y + z = 0$ **1 method for for Gaussian elimination**

$-3y - z = -1.1 \quad (r_2' = r_2 - 2r_1)$

$2y + z = 0.9 \quad (r_3' = r_3 - r_1)$ **1 for this set**

$x + y + z = 0$

$-3y - z = -1.1$

$z = 0.5 \quad (r_3'' = 3r_3 + 2r_2)$ **1 for this equation**

Hence $z = 0.5$; **1 for z**

$y = (1.1 - 0.5)/3 = 0.2$;

$x = -0.2 - 0.5 = -0.7$ **1 for x and y**

Let $u_1, u_2, \ldots, u_n, \ldots$ be an arithmetic sequence and $v_1, v_2, \ldots, v_n, \ldots$ be a geometric sequence. The first terms u_1 and v_1 are both equal to 45, and the third terms u_3 and v_3 are both equal to 5.

(a) Find u_{11}. (3)

(b) Given that $v_1, v_2, \ldots$ is a sequence of **positive** numbers, calculate $\sum_{n=1}^{\infty} v_n$. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	2.4	3		2.4.2		1999 SY1 Q2
(b)	3	2.4	3	(6)	2.4.3		

(a)

$u_3 = 2d + u_1 = 5$ **1 method mark for using formula**

$2d = 5 - 45$

$d = -20$ **1 for d**

$u_{11} = 45 + 10(-20)$

$= -155$ **1 for u_{11}**

(b)

$45r^2 = 5$ **1 method mark for strategy**

$r = \frac{1}{3}$ **1 for value of r**

$S = \frac{45}{1 - \frac{1}{3}}$ **1 for correct substitution**

$= 67\frac{1}{2}$

Differentiate the following functions with respect to x, simplifying your answers where possible.

(a) $h(x) = \sin(x^2)\cos(3x)$. (3)

(b) $y = \dfrac{\ln(x+3)}{(x+3)}, x > -3$. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.2	3		1.2.4		1999 SY1 Q3
(b)	3	1.2	3		1.2.4	1.2.5	

(a)

$$h(x) = \sin(x^2)\cos(3x)$$ **1 method mark (product rule)**

$$h'(x) = 2x\cos(x^2)\cos(3x) - 3\sin(x^2)\sin(3x)$$ **1 for first term**
1 for second term

(b)

$$y = \frac{\ln(x+3)}{(x+3)}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x+3}(x+3) - \ln(x+3).1}{(x+3)^2}$$ **1 method (quotient)**
1 for accuracy

$$= \frac{1 - \ln(x+3)}{(x+3)^2}$$ **1 for simplifying**

Use the substitution $x = 4 \sin t$ to evaluate the definite integral

$$\int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx. \qquad (5)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	2.2		5	2.2.1		1999 SY1 Q5

$$x = 4\sin t \Rightarrow \begin{array}{l} x = 0 \Rightarrow t = 0 \\ x = 2 \Rightarrow t = \frac{\pi}{6} \end{array}$$ **1 for limits**

$$\frac{dx}{dt} = 4\cos t$$ **1**

$$\int_0^2 \frac{x+1}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/6} \frac{4\sin t + 1}{\sqrt{16 - 16\sin^2 t}}\,4\cos t\,dt$$

$$= \int_0^{\pi/6} \frac{4\sin t + 1}{4\cos t}\,4\cos t\,dt$$ **1 for denominator**

$$= \int_0^{\pi/6} (4\sin t + 1)\,dt$$

$$= [-4\cos t + t]_0^{\pi/6}$$ **1**

$$= -2\sqrt{3} + 4 + \frac{\pi}{6}$$ **1**

$$\approx 1{\cdot}059$$

(a) Verify that $z = 2$ is a solution of the equation $z^3 - 8z^2 + 22z - 20 = 0$. (1)

(b) Express $z^3 - 8z^2 + 22z - 20$ as a product of a linear factor and a quadratic factor with real coefficients. Hence find **all** the solutions of $z^3 - 8z^2 + 22z - 20 = 0$. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	1	2.3	1		2.3.9		1999 SY1 Q7
(b)	4	2.3	4		2.3.10		

(a)

$$f(2) = 2^3 - 8 \times 2^2 + 22 \times 2 - 20$$

$$= 8 - 32 + 44 - 20 = 0.$$ **1 for verifying with evidence**

2 is a root

(b)

$$\begin{array}{r|llll} & z^2 & -6z & +10 & \\ z-2 & z^3 & -8z^2 & +22z & -20 \\ & z^3 & -2z^2 & & \\ \hline & & -6z^2 & +22z & \\ & & -6z^2 & +12z & \\ \hline & & & 10z & -20 \\ & & & 10z & -20 \\ \hline \end{array}$$

1 for divisor $z - 2$

1 for getting $z^2 - 6z + 10$

$$z^3 - 8z^2 + 22z - 20 = (z - 2)(z^2 - 6z + 10)$$

$$z^2 - 6z + 10 = 0 \Rightarrow z = \frac{6 \pm \sqrt{36 - 40}}{2}$$ **1**

$$= 3 \pm i$$ **1**

Use integration by parts to obtain

$$\int_0^3 x\sqrt{x + 1}\, dx. \qquad (6)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	6	2.2	6		2.2.2		1999 SY1 Q9

$$\int_0^3 x(x+1)^{\frac{1}{2}}dx = \left[x\int(x+1)^{\frac{1}{2}}dx\right]_0^3 - \int_0^3 1.\tfrac{2}{3}(x+1)^{\frac{3}{2}}dx$$

1 method (parts)

1 for attempting $\int(x+1)^{\frac{1}{2}}dx$

$$= \left[x.\tfrac{2}{3}(x+1)^{\frac{3}{2}}\right]_0^3 - \tfrac{2}{3}.\tfrac{2}{5}\left[(x+1)^{\frac{5}{2}}\right]_0^3$$

1 for $\int(x+1)^{\frac{1}{2}}dx$

1 for $\int(x+1)^{\frac{3}{2}}dx$

$$= \left(2 \times 4^{\frac{3}{2}} - 0\right) - \frac{4}{15}\left(4^{\frac{5}{2}} - 1\right)$$ **1**

$$= 16 - 8\frac{4}{15}$$ **1**

$$= 7\frac{11}{15}$$

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. Write down the matrix $A - \lambda I$, where $\lambda \in \mathbf{R}$ and I is the 3×3 identity matrix.

Find the values of λ for which the determinant of $A - \lambda I$ is zero. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.2		5	3.2.4		1999 SY1 Q11

$$A - \lambda I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & -\lambda & -1 \\ -1 & 1 & -\lambda \end{pmatrix}$$ **1**

$$\det(A - \lambda I) = (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & -\lambda \end{vmatrix}$$ **1 for method**

$$= (1-\lambda)(\lambda^2 + 1) + (\lambda - 1)$$ **1 for accuracy**

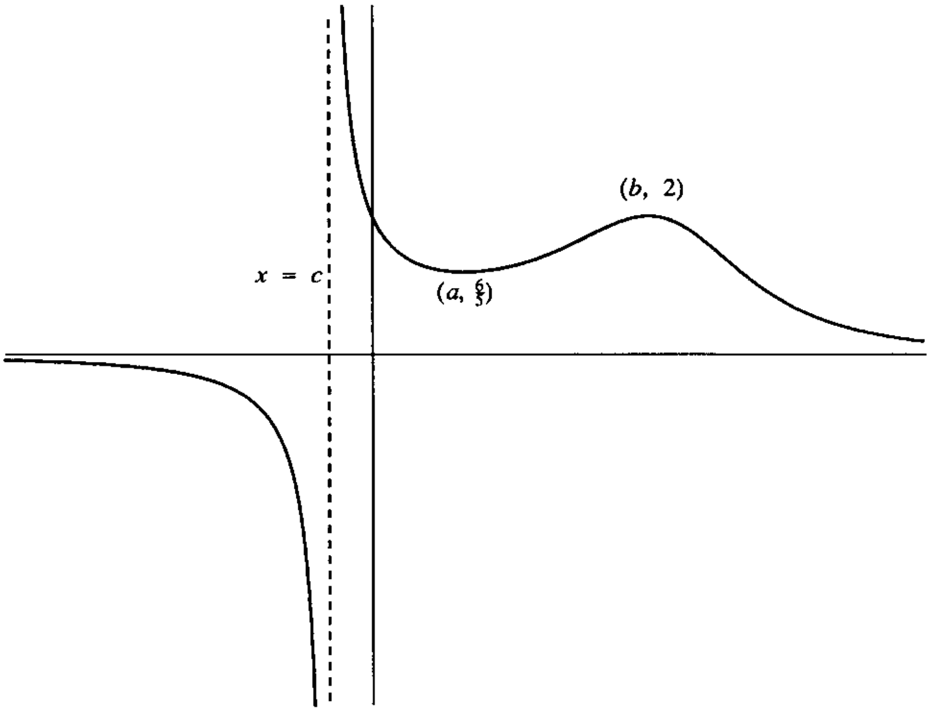
$$= (1-\lambda)(\lambda^2 + 1 - 1) = (1-\lambda)\lambda^2$$

$$(1-\lambda)\lambda^2 = 0$$ **1 for equation**

for $\lambda = 0$ and 1. **1 for answers**

The diagram below shows part of the graph of the function f, where

$$f(x) = \frac{6}{4x^3 - 12x^2 + 9x + 3}.$$



(a) The graph of f has a minimum turning point at $(a, \frac{6}{5})$ and a maximum turning point at $(b,\ 2)$. Use calculus to obtain the values of a and b. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	1.4	4		1.4.9		1999 SY1 Q13

$$f(x) = \frac{6}{4x^3 - 12x^2 + 9x + 3}$$

(a)

$$f'(x) = \frac{-6(12x^2 - 24x + 9)}{(4x^3 - 12x^2 + 9x + 3)^2} = 0 \text{ at S.V.}$$ **1 method** **1 accuracy**

$$3(4x^2 - 8x + 3) = 0$$ **1 for equation**

$$(2x - 1)(2x - 3) = 0$$

i.e. $a = \frac{1}{2};\ b = \frac{3}{2}$ **1 for values**

Let $z = \cos\theta + i\sin\theta$.

(a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ. (5)

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ. (1)

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$. (1)

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where k, m, n, p are integers. State the values of k, m, n, p. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	2.3	5		2.3.6	1.1.4	1999 SY1 Q14
(b)	1	2.3	1		2.3.13		
(c)	1	2.3		1	2.3.3		
(d)	4	2.3		4	2.3.14		

(a) $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta\,(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$

1 method (binomial)

1 accuracy

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

1 for powers of *i*

1 for separating into real and imaginary

Hence the real part is $\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$.

The imaginary part is $(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$ **1**

$= 4\cos\theta\sin\theta(\cos^2\theta - \sin^2\theta)$.

(b) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ **1**

(c) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$ **1**

(d) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

1 for using $\sin^2\theta = 1 - \cos^2\theta$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$ **1**

$= 8\cos^4\theta - 8\cos^2\theta + 1$ **1**

$= 8(\cos^4\theta - \cos^2\theta) + 1$

i.e. $k = 8, m = 4, n = 2, p = 1.$ **1**

In a chemical reaction, two substances X and Y combine to form a third substance Z. Let $Q(t)$ denote the number of grams of Z formed t minutes after the reaction begins. The rate at which $Q(t)$ varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30 - Q)(15 - Q)}{900}.$$

(a) Express $\dfrac{900}{(30 - Q)(15 - Q)}$ in partial fractions. (2)

(b) Use your answer to (a) to show that the general solution of the differential equation can be written in the form

$$A \ln\left(\frac{30 - Q}{15 - Q}\right) = t + C,$$

where A and C are constants.

State the value of A and, given that $Q(0) = 0$, find the value C. (4)

Find, correct to two decimal places,

(i) the time taken to form 5 grams of Z, (1)

(ii) the number of grams of Z formed 45 minutes after the reaction begins. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	1.1	2		1.1.6		1999 SY1 Q15
(b)	4	2.2	4		2.2.5		
(b)(i)	1	2.2		1	2.2.7		
(b)(ii)	2	2.2		2	2.2.7		

(a) $$900 = A(15 - Q) + B(30 - Q)$$ **1 for method**

Letting $Q = 30$ gives $A = -60$

and $Q = 15$ gives $B = 60$ **1 for values**

$$\frac{900}{(30-Q)(15-Q)} = \frac{-60}{(30-Q)} + \frac{60}{(15-Q)}$$

(b) $$\frac{dQ}{dt} = \frac{(30 - Q)(15 - Q)}{900}$$

$$\therefore \quad \int \frac{900}{(30 - Q)(15 - Q)} dQ = \int dt$$ **1 for separating**

$$\therefore \quad \int \frac{-60}{(30 - Q)} + \frac{60}{(15 - Q)} dQ = \int dt$$

$$60 \ln(30 - Q) - 60 \ln(15 - Q) = t + C$$ **1 for integrating**

$$\text{i.e. } 60 \ln\left(\frac{30 - Q}{15 - Q}\right) = t + C$$

$A = 60$ **1**

$C = 60 \ln 2$ (= 41.59 to 2 decimal places) **1**

(i) $$t = 60 \ln\left(\frac{30 - Q}{15 - Q}\right) - 60 \ln 2 = 60 \ln\left(\frac{30 - Q}{2(15 - Q)}\right)$$

When $Q = 5$, $t = 60 \ln \frac{25}{20} = 13.39$ minutes to 2 decimal places. **1**

(ii) $$\ln\left(\frac{30 - Q}{2(15 - Q)}\right) = \frac{t}{60}$$

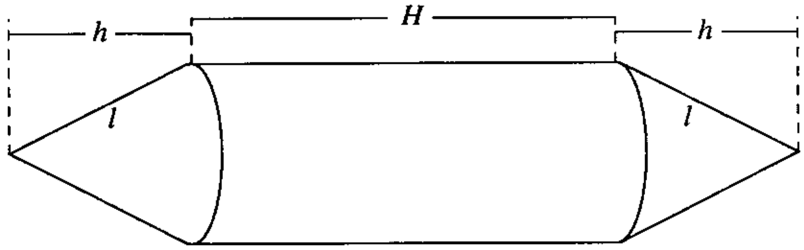
$$30 - Q = 2(15 - Q)e^{t/60}$$ **1 for eliminating logs**

$$Q(2e^{t/60} - 1) = 30(e^{t/60} - 1)$$

$$Q = \frac{30(e^{t/60} - 1)}{2e^{t/60} - 1}$$

When $t = 45$, $Q = 10.36$ grams to 2 decimal places. **1**

A plastic container for holding pens is made from a cylinder with conical ends as shown in the diagram. The cylinder has radius 3 cm and length H cm. Each cone has perpendicular height h cm and slant height l cm. The total volume of the container is 900 cm^3.



(a) Find an expression for H in terms of h. (3)

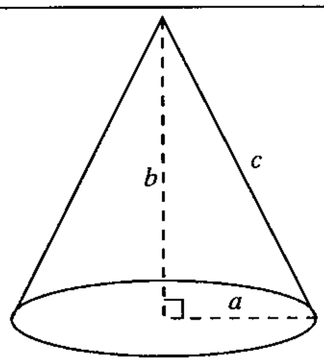
(b) Show that the surface area, S cm^2, of the container is given by

$$S = 600 - 4\pi h + 6\pi\sqrt{9 + h^2}. \quad (3)$$

(c) Find the value of h for which the total surface area of the container is a minimum. Justify your answer. (5)

Note
You may use the following formulae for the volume and curved surface area of a cone with base radius a, perpendicular height b and slant height c.

Volume = $\frac{1}{3}\pi a^2 b$.
Curved surface area = πac.



			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	3	1.2	3		1.2.8		1999 SY1 Q16
(b)	3	1.2.	3		1.2.8		
(c)	3	1.2		3	1.2.8		

(a) $$\pi r^2 H + \tfrac{2}{3}\pi r^2 h = 900$$ **1**

But $r = 3$, $\therefore\ 9\pi H + 6\pi h = 900$ **1**

$$H = \frac{100}{\pi} - \frac{2}{3}h$$ **1**

(b) Area of curved surface of one cone $= \pi r l = 3\pi\sqrt{h^2+9}$ **1 for using $l = \sqrt{h^2+9}$**

Curved surface area of the cylinder

$$= 2\pi r H = 6\pi\left(\frac{100}{\pi} - \frac{2}{3}h\right) = 600 - 4\pi h$$ **1**

So, $S = 600 - 4\pi h + 6\pi\sqrt{h^2 + 9}$ **1**

(c) $$\frac{dS}{dh} = -4\pi + 6\pi.\tfrac{1}{2}.2h\,(h^2 + 9)^{-\frac{1}{2}}$$ **1**

$$= -4\pi + \frac{6\pi h}{\sqrt{h^2 + 9}} = 0 \text{ at stationary values}$$ **1**

$$3h = 2\sqrt{h^2 + 9}$$ **1 for this (or equivalent)**

$$9h^2 = 4h^2 + 36$$

$$h^2 = \frac{36}{5}$$

$$h = \frac{6}{\sqrt{5}} = \frac{6}{5}\sqrt{5}$$ **1 for this (or equivalent)**

$$\frac{dS}{dh} = -4\pi + \frac{6\pi h}{\sqrt{h^2 + 9}}$$

$$\frac{d^2S}{dh^2} = 6\pi\frac{(h^2 + 9)^{1/2} - h\left(\tfrac{1}{2}\right)2h\,(h^2 + 9)^{-1/2}}{h^2 + 9}$$

$$= 6\pi\frac{h^2 + 9 - h^2}{(h^2 + 9)^{3/2}} = \frac{54\pi}{(h^2 + 9)^{3/2}} > 0$$

So the stationary value is a minimum turning point. **1 for checking nature**

Use induction to prove that

$$\sum_{r=1}^{n} r(r+1) = \tfrac{1}{3}n(n+1)(n+2)$$

for all positive integers n. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.5		5	3.5.6		1999 SY2 Q1

True for $n = 1$ since LHS $= 1 \times 2 = 2$ and RHS $= \frac{1}{3} \times 1 \times 2 \times 3 = 2$. **1**

Suppose true for $n = k$, so that

$$\sum_{r=1}^{k} r(r+1) = \tfrac{1}{3}k(k+1)(k+2) \qquad \mathbf{1}$$

Then

$$\sum_{r=1}^{k+1} r(r+1) = \sum_{r=1}^{k} r(r+1) + (k+1)(k+2)$$

$$= \tfrac{1}{3}k(k+1)(k+2) + (k+1)(k+2) \qquad \mathbf{1}$$

$$= \tfrac{1}{3}(k+1)(k+2)(k+3) \qquad \mathbf{1}$$

$$= \tfrac{1}{3}(k+1)((k+1)+1)((k+1)+2) \qquad \mathbf{1}$$

So the result is true for $n = k + 1$.

Since it is true for $n = 1$ and true for $n = k + 1$ whenever it is true for $n = k$, it is true for all $n \geqslant 1$.

The $n \times n$ matrix A satisfies the equation

$$A^2 = 5A + 3I$$

where I is the $n \times n$ identity matrix. Show that A is invertible and express A^{-1} in the form $pA + qI$. (2)

Obtain a similar expression for A^4. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	3.2	2	2	3.2.2		1999 SY2 Q2

$A(A - 5I) = 3I$ **1**

So $A \times \frac{1}{3}(A - 3I) = I$, so that A^{-1} exists and is $\frac{1}{3}(A - 5I)$. **1**

$$A^4 = A^2 \times A^2 = (5A + 3I)(5A + 3I)$$

$$= 25A^2 + 30A + 9I$$ **1**

$$= 25(5A + 3I) + 30A + 9I$$

$$= 155A + 84I$$ **1**

(i) The sequence $\{u_n\}$ is defined by $u_1 = 1, \quad u_n = 3u_{n-1} + 2 \ (n \geqslant 2)$. Let

$$A_n = \begin{pmatrix} u_n & 1 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}.$$

Show that $A_{n-1}B = A_n$, and deduce that

$$A_n = A_1 B^{n-1}. \qquad (4)$$

By taking determinants, deduce that $u_n = 2 \times 3^{n-1} - 1$. (2)

(ii) Let $S = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ and $T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

(a) Show that $D = T'ST$ is a diagonal matrix. (2)

(b) Explain briefly the geometrical significance of the transformation $\mathbf{x} = T\mathbf{X}$ where $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$. (1)

(c) Use this transformation to rewrite the equation

$$5x^2 + 6xy + 5y^2 = 16$$

in terms of X and Y. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	6	3.2	4	2	3.2.2	3.2.4	1999 SY2 Q7
(ii)	6	3.2	3	3	3.2.2	3.2.9	

(a)

$$A_{n-1}B = \begin{pmatrix} u_{n-1} & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix} \qquad \mathbf{1}$$

$$= \begin{pmatrix} 3u_n + 2 & 1 \\ -1 & 1 \end{pmatrix} = A_n \qquad \mathbf{1}$$

So $A_n = A_{n-1}B = A_{n-2}B^2 = A_{n-3}B^3 = \ldots = A_1B^{n-1}$. **2**

Thus $|A_n| = |A_1| \times |B|^{n-1}$. **1**

i.e. $u_{n+1} = 2 \times 3^{n-1}$. **1**

(b) (i)

$$D = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 2 & -2 \\ 8 & 8 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \mathbf{1}$$

$$= \frac{1}{2}\begin{pmatrix} 4 & 0 \\ 0 & 16 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \qquad \mathbf{1}$$

(ii) Rotation of axes through $\frac{\pi}{4}$. **1**

(iii)

$$5x^2 + 6xy + 5y^2 = \mathbf{x}'S\mathbf{x} \qquad \mathbf{1}$$

$$= \mathbf{x}'T'ST\mathbf{x} \qquad \mathbf{1}$$

$$= \mathbf{x}'D\mathbf{x} = 2X^2 + 8Y^2 \qquad \mathbf{1}$$

i.e. equation becomes $2X^2 + 8Y^2 = 16$.

(i) (a) Use the Euclidean algorithm to find integers x and y such that

$$37x + 23y = 1.$$ (3)

(b) **It is given** that if $x = x_0$, $y = y_0$ is a particular integer solution of the equation $ax + by = c$, where a, b, c are integers with a and b coprime, then the general integer solution is $x = x_0 - bt$, $y = y_0 + at, t \in \mathbb{Z}$.
Hence solve the following problem.

Chocobars cost 23 pence each, and Choconuts cost 37 pence each. Jo bought some of each, and the total cost was exactly £10. How many Chocobars did Jo buy? (6)

(ii) Use the method of proof by contradiction to show that $\sqrt{3}$ is irrational. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)(a)	3	3.5	3		3.5.10		1999 SY2 Q8
(b)	6	3.5		6	3.5.11		
(ii)	4	2.5	4		2.5.6		

(a)

$$37 = 1 \times 23 + 14$$
$$23 = 1 \times 14 + 9$$
$$14 = 1 \times 9 + 5$$
$$9 = 1 \times 5 + 4$$
$$5 = 1 \times 4 + 1 \qquad \mathbf{1}$$

So

$$1 = 5 - (9 - 5) = 2 \times 5 - 9 \qquad \mathbf{1}$$
$$= 2(14 - 9) - 9 = 2 \times 14 - 3 \times 9$$
$$= 2 \times 14 - 3(23 - 14) = 5 \times 14 - 3 \times 23$$
$$= 5(37 - 23) - 3 \times 23$$
$$= 5 \times 37 - 8 \times 23 \qquad \mathbf{1}$$

Thus $x = 5$ and $y = -7$.

Let x = number of choconuts and y = number of chocobars.
So, we have to solve

$$37x + 23y = 1000 \qquad \mathbf{1}$$

One solution is $x_0 = 5000, y_0 = -8000$ **1**
so the general solution is

$$x = 5000 - 23t, \qquad y = -8000 + 37t \qquad \mathbf{1}$$

Require $t \leqslant \dfrac{5000}{23} = 217{\cdot}39\ldots$

and $t \geqslant \dfrac{8000}{37} = 216{\cdot}21\ldots$ **1**

Thus $t = 217$ **1**

which gives $y = 37 \times 217 - 8000 = 29.$ **1**

(b) Suppose $\sqrt{3} = \dfrac{a}{b}$ where a, b are coprime integers. **1**

Then $a^2 = 3b^2$ so $3 \mid a^2$ so $3 \mid a$. **1**

So let $a = 3c$ and the equation becomes

$$3b^2 = 9c^2 \Rightarrow b^2 = 3c^2 \qquad \mathbf{1}$$

But then $3 \mid b$ and so a, b are not coprime. Thus a contradiction is established and hence $\sqrt{3}$ is irrational. **1**

(a) Show that if $M = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ then $M^2 = I$ where I is the 2×2 identity matrix. (1)

By choosing two different values of θ, exhibit two matrices A, B such that $A^2 = I$ and $B^2 = I$ but $(AB)^2 \neq I$. (4)

(b) Prove that if C and D are $n \times n$ matrices such that $C^2 = I$, $D^2 = I$ **and C and D commute**, then $(CD)^2 = I$. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	3.2	1	4	3.2.2		1999 SY2 Q9
(b)	2	3.2	2		3.2.3		

(a)
$$M^2 = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{pmatrix} \quad \mathbf{1}$$
$$= I$$

Take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. **2**

Then $AB = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **1**

where $(AB)^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \neq I.$ **1**

(b) Suppose $C^2 = D^2 = I$ and $CD = DC$.

Then

$$(CD)^2 = CD.CD \quad \mathbf{1}$$
$$= DC.CD$$
$$= D.C^2.D = DD = I \quad \mathbf{1}$$

(i) Show that the lines

$$L_1 : \frac{x-3}{2} = \frac{y+1}{3} = \frac{z-6}{1}$$

$$L_2 : \frac{x-3}{-1} = \frac{y-6}{2} = \frac{z-11}{2}$$

intersect, and find the point of intersection. (6)

Obtain an equation for the plane containing both L_1 and L_2. (3)

(ii) Which of the following statements about 3-dimensional vectors are true and which are false? Justify each true statement and give a counter-example to each false statement.

(a) $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$ for all **u**, **v**.

(b) $(\mathbf{u} \times \mathbf{v}).\mathbf{u} = 0$ for all **u**, **v**.

(c) $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = \mathbf{v}.(\mathbf{u} \times \mathbf{w})$ for all **u**, **v**, **w**. (6)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	6	3.1	6		3.1.6		SY2 1999 Q10
	3	3.1		3	3.1.7		
(ii)	6	3.5	6		3.5.4	3.1.3/4	

(i) $L_1 : x = 2s + 3, y = 3s - 1, z = s + 6$ **1**

$L_2 : x = -t + 8, y = -2t + 6, z = 2t + 11$ **1**

Equating the x and y formulas gives

$$2s + 3 = -t + 8 \text{ and } 3s - 1 = -2t + 6$$ **1**

$$\Rightarrow t = -2s + 5 \text{ and } 3s = -2s + 7$$

$$\text{leading to } s = 3 \text{ and } t = -1$$ **1**

Checking z: $s + 6 = 9$ and $2t + 11 = 9$ **1**

so the lines *do* intersect.

Point of intersection is (9, 8, 9). **1**

Normal to plane is

$$(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (-\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$ **1**

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ -1 & -2 & 2 \end{vmatrix} = 8\mathbf{i} - 5\mathbf{j} - \mathbf{k}$$ **1**

So the equation of the plane is

$$8x - 5y - z = 72 - 40 - 9$$

$$8x - 5y - z = 23$$ **1**

(ii) (a) False **1**

e.g. take $\mathbf{u} = \mathbf{i}, \mathbf{v} = \mathbf{j}$ **1**

(b) True

$\mathbf{u} \times \mathbf{v}$ is perpendicular to $\mathbf{u}$ **1**

so $(\mathbf{u} \times \mathbf{v}).\mathbf{u} = 0$. **1**

(c) False **1**

e.g. $\mathbf{i}.(\mathbf{j} \times \mathbf{k}) = \mathbf{i}.\mathbf{i} = 1, \mathbf{j}.(\mathbf{i} \times \mathbf{k}) = \mathbf{j}.(-\mathbf{j}) = -1$ **1**

Note: in (a) and (b) an unsupported claim of 'false' would not get a mark.

(a) Differentiate $f(x) = e^{x^2+3}$. (2)

(b) Differentiate $g(x) = \ln\sqrt{x^2 + 3}$.

Hence find $\int \frac{5x}{x^2 + 3}\,dx$. (3, 2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	1.2	2		1.2.4	1.2.5	2000 SY1 Q1
(b)	3	2.1		3	2.1.5		
	2	2.2		2	2.2.1		

(a)

$f(x) = e^{x^2+3} \Rightarrow f'(x) = 2x\,e^{x^2+3}$ **1 for $2x$** **1 for e^{x^2+3}**

(b) **Method 1**

$g(x) = \ln\sqrt{x^2 + 3} = \frac{1}{2}\ln(x^2 + 3)$ **1 for simplifying**

$g'(x) = \frac{1}{2}\frac{2x}{(x^2 + 3)} = \frac{x}{(x^2 + 3)}$. **1 for $2x$** **1 for $(x^2 + 3)$**

Method 2

$g(x) = \ln\sqrt{x^2 + 3} = \ln(x^2 + 3)^{1/2}$

$g'(x) = \frac{\frac{1}{2}(x^2 + 3)^{-\frac{1}{2}}\,2x}{(x^2 + 3)^{\frac{1}{2}}} = \frac{x}{(x^2 + 3)}$. **1 for $\frac{1}{2}(x^2 + 3)^{-\frac{1}{2}}$** **1 for $2x$** **1 for $(x^2 + 3)^{\frac{1}{2}}$**

(c)

$\int \frac{5x}{(x^2 + 3)}\,dx = 5\int \frac{x}{(x^2 + 3)}\,dx$ **1 for handling the '5'**

$= 5\ln\sqrt{x^2 + 3} + c$ **1 for result**

Let A be the matrix

$$A = \begin{pmatrix} -1 & 6 & -3 \\ -2 & 7 & -3 \\ -4 & 12 & -5 \end{pmatrix}.$$

Show that $A^3 + A^2 - A = I$ where I denotes the 3×3 identity matrix. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	3.4	4		3.4.2		2000 SY1 Q3

$$A^2 = \begin{pmatrix} -1 & 6 & -3 \\ -2 & 7 & -3 \\ -4 & 12 & -5 \end{pmatrix}\begin{pmatrix} -1 & 6 & -3 \\ -2 & 7 & -3 \\ -4 & 12 & -5 \end{pmatrix}$$ **1 method for multiplying**

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ **2 E 1 for accuracy**

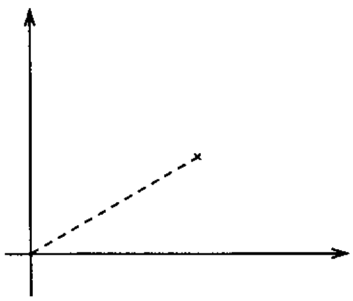
$\therefore A^3 + A^2 - A = A + I - A = I.$ **1 for evidence of $A^3 = A$, etc**

Plot the complex number $z = \sqrt{3} + i$ on an Argand diagram and find the modulus and argument of z. (3)

Calculate $\frac{\bar{z}}{z}$ where $\bar{z}$ denotes the complex conjugate of z. (2)

Use de Moivre's theorem to evaluate z^6. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	2.3	3		2.3.7	2.3.4	2000 SY1 Q4
	2	2.3	2		2.3.6	2.3.7	
	2	2.3	2		2.3.13		



1 for reasonable diagram

$$|z| = \sqrt{3+1} = 2$$ **1**

$$\arg z = \tan^{-1}\frac{1}{\sqrt{3}} = 30°$$ **1**

$$\frac{\bar{z}}{z} = \frac{\sqrt{3} - i}{\sqrt{3} + i} \times \frac{\sqrt{3} - i}{\sqrt{3} - i}$$ **1**

$$= \frac{2 - 2\sqrt{3}i}{4} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$ **1**

$$z^6 = 2^6(\cos 6 \times 30° + i \sin 6 \times 30°)$$ **1 for using de Moivre**

$$= 2^6 \times (-1) = -64.$$ **1**

The function g is defined by

$$g(x) = x\tan^{-1}x, \qquad x \in \mathbf{R}.$$

Verify that the second derivative of g is given by

$$g''(x) = \frac{C}{(1 + x^2)^2}$$

where C is a constant. State the value of C. (5)

Explain why the graph of g has no points of inflexion. (1)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
		2.1	2		2.1.1	1.2.4	2000 SY1 Q6
		1.2	3		1.2.4		
		1.4	1		1.4.3		

$$g(x) = x\tan^{-1}x$$

$$\therefore g'(x) = \tan^{-1}x + \frac{x}{x^2 + 1}$$ **1 for using product rule**
1 for $\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2}$

$$g''(x) = \frac{1}{x^2 + 1} + \frac{(x^2 + 1) - x.2x}{(x^2 + 1)^2}$$ **1 for quotient rule**

$$= \frac{x^2 + 1 + 1 - x^2}{(x^2 + 1)^2}$$ **1**

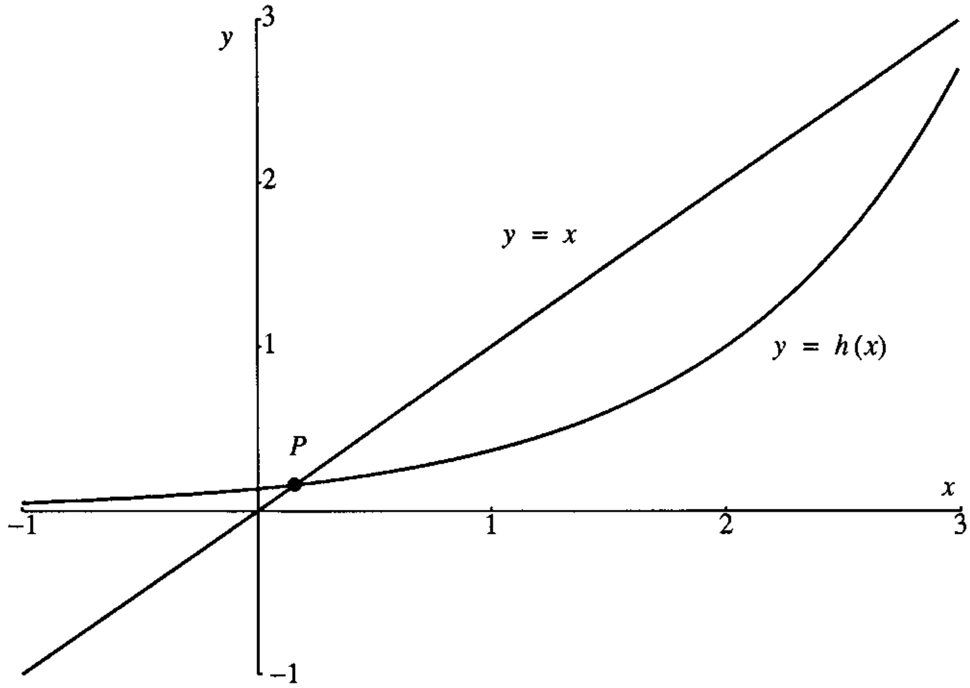
$$= \frac{2}{(x^2 + 1)^2}$$ **1**

(therefore $C = 2$).

$g''(x) > 0$ (or $g''(x) \neq 0$) so the graph of g has no points of inflexion (or equivalent). **1**

The diagram below shows the graphs of $y = x$ and $y = h(x)$ where the function h is defined by

$$h(x) = e^{x-2}, \qquad -1 \leqslant x \leqslant 3.$$

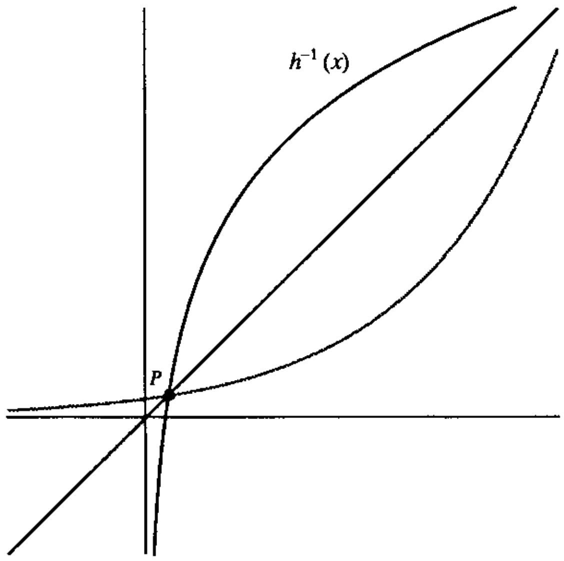


(i) Sketch the graph of the inverse function h^{-1}.

(ii) Find a formula for h^{-1}. (1, 2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(i)	1	1.4	1		1.4.7		2000 SY1 Q7
(ii)	2	1.4	2		1.4.7		

(i)



1

(ii) Let $y = e^{x-2}$ then $x - 2 = \ln y$ 1

i.e. $x = 2 + \ln y$

i.e. $h^{-1}(x) = 2 + \ln x$ 1

The acceleration of a particle travelling in a straight line is given by $\frac{1}{1+t^2}$ ms^{-2}, where t is the time in seconds since the particle started moving. Given that the velocity is zero when $t = 1$ find the velocity when $t = \sqrt{3}$. (4)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	4	1.2	4		1.2.8		2000 SY1 Q8

$$\frac{dv}{dt} = \frac{1}{1+t^2}$$ **1**

$$v = \int \frac{1}{1+t^2} dt$$

$$= \tan^{-1} t + c$$ **1**

$$t = 1 \Rightarrow v = 0 \Rightarrow c = -\frac{\pi}{4}$$ **1**

$$t = \sqrt{3} \Rightarrow v = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12} \approx 0.262$$ **1**

Express in partial fractions

$$\frac{11 - 2x}{x^2 + x - 2}. \qquad (3)$$

Hence obtain

$$\int_3^5 \frac{11 - 2x}{x^2 + x - 2}dx \qquad (3)$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	1.1			1.1.6		2000 SY1 Q10
	3	2.2			2.2.1		

$$\frac{11 - 2x}{x^2 + x - 2} = \frac{11 - 2x}{(x - 1)(x + 2)}$$

$$= \frac{A}{x - 1} + \frac{B}{x + 2}$$

$$11 - 2x = A(x + 2) + B(x - 1) \qquad 1$$

$$x = 1 \Rightarrow 3A = 9; A = 3 \qquad 1$$

$$x = -2 \Rightarrow -3B = 15; B = -5 \qquad 1$$

$$\int_3^5 \frac{11 - 2x}{x^2 + x - 2}dx = \int_3^5 \frac{3}{x - 1} - \frac{5}{x + 2}dx \qquad 1$$

$$= [3\ln(x - 1) - 5\ln(x + 2)]_3^5 \qquad 1$$

$$= 3\ln 4 - 3\ln 2 - 5\ln 7 + 5\ln 5 \qquad 1$$

$$= \left(\ln\frac{2^3 5^5}{7^5}\right) \approx 0.397.$$

The function f is defined by

$$f(x) = e^x \cos x, \qquad 0 \leqslant x \leqslant \tfrac{\pi}{2}.$$

(a) Find the stationary point of f and determine its nature.
Sketch the graph of f showing clearly where the graph meets the x-axis and the y-axis. (5)

(b) By integrating by parts **twice**, show that

$$\int f(x)\,dx = \tfrac{1}{2}e^x(\cos x + \sin x) + C$$

where C denotes the constant of integration. (4)

(c) Use the previous results to calculate the area bounded by the graph of f and the coordinate axes. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	1.4	5		1.4.9	1.2.4/5	2000 SY1 Q12
(b)	4	2.2	2	2	2.2.4	1.3.3	
(c)	2			2	1.3.6		

(a)

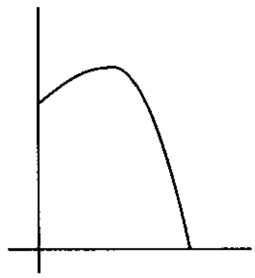
$$f'(x) = e^x \cos x - e^x \sin x = 0$$ **1 for product rule**
1 for accuracy

when $\tan x = 1$ i.e. $x = \frac{\pi}{4}$ **1**

$f'(x) = e^x(\cos x - \sin x) > 0$ for $x < \frac{\pi}{4}$

$f'(x) = e^x(\cos x - \sin x) < 0$ for $x > \frac{\pi}{4}$

i.e. $(\frac{\pi}{4}, e^{\pi/4}/\sqrt{2})$ is a maximum. **1 for justifying**



1

(b)

$$\int f(x)\,dx = \int e^x \cos x\,dx$$

$$= e^x \int \cos x\,dx - \int e^x \int \cos x\,dx\;dx$$ **1**

$$= e^x \sin x - \int e^x \sin x\,dx$$ **1**

$$= e^x \sin x - \left(e^x \int \sin x\,dx - \int e^x(-\cos x)\,dx\right)$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx$$ **1**

$$\therefore 2\int e^x \cos x\,dx = e^x(\sin x + \cos x) + C'$$ **1**

i.e. $\int f(x)dx = \frac{1}{2}e^x(\cos x + \sin x) + C$

(c)

$$\text{Area} = \int_0^{\pi/2} f(x)\,dx$$

$$= \left[\frac{1}{2}e^x(\cos x + \sin x)\right]_0^{\pi/2}$$ **1**

$$= \frac{1}{2}\left(e^{\pi/2} - 1\right)(\approx 1.91)$$ **1**

A car manufacturer is planning future production patterns. Based on estimates of time, cost and labour, he obtains a set of three equations for the numbers x, y, z of three new types of car. These equations are

$$x + 2y + z = 60$$

$$2x + 3y + z = 85$$

$$3x + y + (\lambda + 2)z = 105,$$

where the **integer** λ is a parameter such that $0 < \lambda < 10$.

(a) Use Gaussian elimination to find an expression for z in terms of λ. (5)

(b) Given that z must be a positive integer, what are the possible values for z? (2)

(c) Find the corresponding values of x and y for each value of z. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	5	1.5		5	1.5.4		2000 SY1 Q13
(b)	2						
(c)	2						

(a)

$$\begin{aligned} x + 2y + z &= 60 \\ 2x + 3y + z &= 85 \\ 3x + y + (\lambda + 2)z &= 105 \end{aligned}$$

$$\begin{aligned} x + 2y + z &= 60 \\ -y - z &= -35 \\ -5y + (\lambda - 1)z &= -75 \end{aligned}$$

1 for using Gaussian elimination
1 for new equation 2
1 for new equation 3

$$\begin{aligned} x + 2y + z &= 60 \\ -y - z &= -35 \\ (4 + \lambda)z &= 100 \end{aligned}$$

1 for eliminating *y*

so $z = \dfrac{100}{4 + \lambda}$ **1**

(b) $4 + \lambda$ is a factor of 100,
so $\lambda = 1$ which gives $z = 20$ **1**
or $\lambda = 6$ giving $z = 10$. **1**

(or give 1 mark for both values of λ)

(c) $z = 20 \Rightarrow y = 15; x = 10$ **1**
$z = 10 \Rightarrow y = 25; x = 0.$ **1**

By writing

$$(k + 1)^3 - k^3 = 3k^2 + 3k + 1,$$

show that

$$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}n(n + 1)(2n + 1). \qquad (5)$$

Deduce that

$$2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = \tfrac{2}{3}n(n + 1)(2n + 1). \qquad (2)$$

Hence obtain the sum of the squares of all the even integers between 99 and 201. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.5	5		3.5.6		2000 SY1 Q14
	4	3.5		4	3.5.8		

$$\sum_{k=1}^{n}\left((k + 1)^3 - k^3\right) = \sum_{k=1}^{n}\left(3k^2 + 3k + 1\right).$$ **1 for idea of summation**

$$\text{So } (n + 1)^3 - 1 = 3\sum k^2 + 3\sum k + n$$ **1 for 'telescoping'** **1 for RHS**

$$3\sum k^2 = (n+1)^3 - \tfrac{3}{2}n(n+1) - (n+1)$$ **1 for using $\sum k = \frac{1}{2}n(n+1)$**

$$= (n+1)\left(n^2 + 2n + 1 - \frac{3}{2}n - 1\right)$$

$$= \tfrac{1}{2}(n+1)(2n^2 + n)$$ **1**

$$\text{i.e.} \sum k^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$2^2 + 4^2 + 6^2 + \ldots + (2n)^2 = 2^2\left(1^2 + 2^2 + \ldots + n^2\right)$$ **1 for common factor**

$$= \tfrac{4}{6}n(n + 1)(2n + 1)$$ **1 for using result above**

$$= \tfrac{2}{3}n(n + 1)(2n + 1).$$

$$100^2 + \ldots + 200^2 = \tfrac{2}{3}\left(100 \times 101 \times 201 - 49 \times 50 \times 99\right)$$ **1 for using 100 and 49**

$$= 1191700$$ **1**

A tank initially holds 20 litres of pure water. A solution of water containing 0.1 kg of salt per litre flows into the tank at a rate of 4 litres per minute. The contents of the tank are stirred continually to maintain a uniform concentration and liquid flows out at the same rate. At time t minutes, the water in the tank contains x kg of salt.

(a) Write down expressions for
   (i) the amount of salt flowing into the tank per minute, (1)
   (ii) the amount of salt flowing out of the tank per minute. (1)

   Hence show that at any time $t > 0$, the amount of salt, x kg, in the tank can be modelled by the differential equation

$$\frac{dx}{dt} = \frac{2 - x}{5} \quad (2)$$

(b) Find a formula for x in terms of t. (4)

(c) How much salt is present after 20 minutes? (2)

(d) In the long term, what will be the amount of salt in the tank? (1)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	2.2	4		2.2.6		2000 SY1 Q15
(b)	4	2.2	4		2.2.5		
(c)	2						
(d)	1						

(a) (i) the amount of salt flowing in per minute is 0.4 kg **1**

(ii) the amount of salt flowing out per minute is $\frac{4x}{20} = \frac{x}{5}$. **1**

$$\frac{dx}{dt} = 0.4 - \frac{x}{5} = \frac{2 - x}{5} \quad \mathbf{1}$$

(b)

$$\int \frac{dx}{2 - x} = \int \frac{dt}{5} \quad \mathbf{1}$$

$$-\ln(2 - x) = \frac{t}{5} + c \quad \mathbf{1}$$

$$t = 0, x = 0 \Rightarrow c = -\ln 2 \quad \mathbf{1}$$

$$\frac{2}{2 - x} = e^{t/5}$$

$$x = 2\left(1 - e^{-t/5}\right) \quad \mathbf{1}$$

(c)

$$t = 20 \Rightarrow x = 2\left(1 - e^{-4}\right) \quad \mathbf{1}$$

$$x \approx 1{\cdot}96 \text{ kg} \quad \mathbf{1}$$

(d) In the long term, $x \to 2$ so the limit is 2 kg. **1**

Use the Euclidean Algorithm to find integers x, y such that

$$181x + 79y = 1.$$

Hence write down the multiplicative inverse of 79 in $\mathbb{Z}_{181}$. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.5	2	3	3.5.10	3.5.11	2000 SY2 Q1

$$181 = 2 \times 79 + 23$$ **1**

$$79 = 3 \times 23 + 10$$

$$23 = 2 \times 10 + 3$$

$$10 = 3 \times 3 + 1$$ **1**

Hence

$$1 = 10 - 3(23 - 2 \times 10)$$ **1**

$$= 7 \times 10 - 3 \times 23$$

$$= 7(79 - 3 \times 23) - 3 \times 23$$

$$= 55 \times 79 - 24 \times 181$$ **1**

The inverse of 79 is 55. **1**

Use induction to prove that

$$\sum_{r=1}^{n} \frac{1}{3^r} = \frac{1}{2}\left(1 - \frac{1}{3^n}\right)$$

for all positive integers n. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	5	3.5	1	4	3.5.6		2000 SY2 Q3

When $n = 1$, LHS $= \sum_{r=1}^{1} \frac{1}{3^r} = \frac{1}{3}$; RHS $= \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$; so true for $n = 1$. **1**

Assume true for $n = k$, so that $\sum_{r=1}^{k} \frac{1}{3^r} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right)$. **1**

Then $\sum_{r=1}^{k+1} \frac{1}{3^r} = \sum_{r=1}^{k} \frac{1}{3^r} + \frac{1}{3^{k+1}} = \frac{1}{2}\left(1 - \frac{1}{3^k}\right) + \frac{1}{3^{k+1}}$ **1**

$= \frac{1}{2}\left(1 - \frac{3}{3^{k+1}} + \frac{2}{3^{k+1}}\right)$ **1**

$= \frac{1}{2}\left(1 - \frac{1}{3^{k+1}}\right)$.

So the result is true for $n = k + 1$.

Since it is true for $n = 1$ and true for $n = k + 1$ whenever it is true for $n = k$, it is true for all $n \geqslant 1$. **1**

A square matrix M is *orthogonal* if $M'M = I$.

(a) Prove that if A and B are $n \times n$ orthogonal matrices then so is AB. (2)

(b) Prove that if A is orthogonal then $\det A = \pm 1$. (2)
(You may assume that $\det A = \det A'$.)

(c) Give an example of a 2×2 matrix P which is not orthogonal but is such that $\det P = \pm 1$. Justify your answer. (2)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	2	3.2	2		3.2.3		2000 SY2 Q4
(b)	2	3.2	2		3.2.4		
(c)	2	3.2		2	3.2.4	3.5.4	

(a) Given $AA' = BB' = I$, then

$$(AB)'(AB) = B'A'AB \qquad \mathbf{1}$$
$$= B'(A'A)B = B'IB$$
$$= B'B = I \qquad \mathbf{1}$$

(b)
$$\det(A'A) = \det I = 1 \qquad \mathbf{1}$$
$$\det A' . \det A = 1$$
$$\therefore (\det A)^2 = 1 \Rightarrow \det A = \pm 1 \qquad \mathbf{1}$$

(c) Any appropriate example will do.

For example, $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is such that $\det P = 1 \times 1 - 0 \times 1 = 1$. **1**

But $P'P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq I$. **1**

Consider the following two statements S and T.
S: If p and q are two odd prime numbers then $p + q$ is not prime.
T: If p and q are two odd prime numbers then $p - q$ is not prime.
For each of S and T, give a proof if it is true, or give a counter-example if it is false. (3)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
	3	3.5	3		3.5.3	3.5.4	2000 SY2 Q5

S is true. If p and q are two odd primes then $p + q$ is even. **1**
Since odd primes are greater than or equal to 3, $p + q$ cannot be 2. **1**

T is false. For example $p = 5, q = 3$. **1**
(Other examples will do, but they *must* differ by 2.)

Throughout this question

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}.$$

(a) Find A^{-1}, M^3 and M^3A. (4)

(b) The Fibonacci numbers f_n are defined by

$$f_1 = 1, \; f_2 = 2, \; f_n = f_{n-1} + f_{n-2} \quad (n \geqslant 3).$$

Thus, each Fibonacci number is the sum of the previous two:

$$f_3 = f_2 + f_1 = 2 + 1 = 3, \; f_4 = 3 + 2 = 5, \quad \text{etc.}$$

Prove by induction that

$$M^{n+2} = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix}$$

for each positive integer n. (4)

(c) The Lucas numbers λ_n are defined by

$$\lambda_1 = 1, \; \lambda_2 = 3, \; \lambda_n = \lambda_{n-1} + \lambda_{n-2} \quad (n \geqslant 3),$$

so that $\lambda_3 = 3 + 1 = 4$, $\lambda_4 = 4 + 3 = 7$, etc.

Verify that

$$M^3A = \begin{pmatrix} \lambda_4 & \lambda_5 \\ \lambda_5 & \lambda_6 \end{pmatrix}.$$ (2)

(d) It is given that $M^nA = \begin{pmatrix} \lambda_{n+1} & \lambda_{n+2} \\ \lambda_{n+2} & \lambda_{n+3} \end{pmatrix}$ for all positive integers n. By using the identity $M^n = (M^nA)A^{-1}$, and the results of (a) and (b), show that f_n can be expressed in the form

$$f_n = s\lambda_{n+2} - t\lambda_{n+3}$$

and determine the values of s and t. (5)

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	3.2	4		3.2.4	3.2.2	2000 SY2 Q7
(b)	5	3.5	5		3.5.6	3.2.2	
(c)	2	3.5	2		3.5.3		
(d)	4	3.2		4	3.2.2		

(a)

$$A^{-1} = \frac{1}{4-9}\begin{pmatrix} 4 & -3 \\ -3 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -4 & 3 \\ 3 & -1 \end{pmatrix}$$ **2**

$$M^3 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$ **1**

$$M^3A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 11 \\ 11 & 18 \end{pmatrix}$$ **1**

(b) When $n = 1$, LHS $= M^3 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ (from (a)) and

RHS $= \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ so result holds when $n = 1$. **1**

Suppose the result is true for $n = k$, i.e. $M^{k+2} = \begin{pmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{pmatrix}$. Then **1**

$$M^{k+3} = M^{k+2}M = \begin{pmatrix} f_k & f_{k+1} \\ f_{k+1} & f_{k+2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$ **1**

$$= \begin{pmatrix} f_{k+1} & f_k + f_{k+1} \\ f_{k+2} & f_{k+1} + f_{k+2} \end{pmatrix} = \begin{pmatrix} f_{k+1} & f_{k+2} \\ f_{k+2} & f_{k+3} \end{pmatrix}$$ **1**

$$= \begin{pmatrix} f_{k+1} & f_{(k+1)+1} \\ f_{(k+1)+1} & f_{(k+1)+2} \end{pmatrix}$$ **1**

Since result is true for $n = 1$ and true for $n = k + 1$ whenever it is true for $n = k$, it is true for all $n \geqslant 1$. **1**

(c) $\lambda_5 = 7 + 4 = 11, \lambda_6 = 11 + 7 = 18.$ **1**

and $M^3A = \begin{pmatrix} 7 & 11 \\ 11 & 18 \end{pmatrix} = \begin{pmatrix} \lambda_4 & \lambda_5 \\ \lambda_5 & \lambda_6 \end{pmatrix}$. **1**

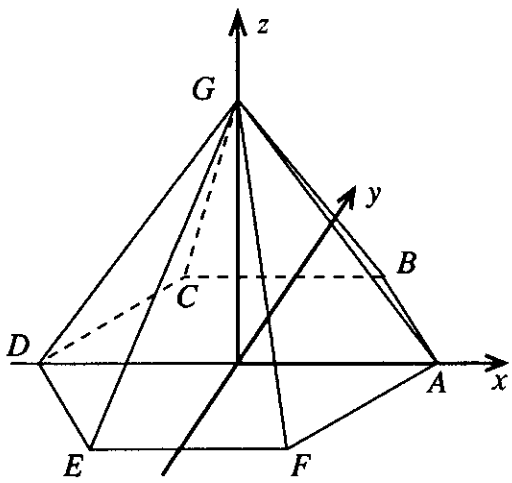
(d) $M^n = (M^nA)A^{-1}$ becomes

$$\begin{pmatrix} f_{n-2} & f_{n-1} \\ f_{n-1} & f_n \end{pmatrix} = \frac{1}{5}\begin{pmatrix} \lambda_{n+1} & \lambda_{n+2} \\ \lambda_{n+2} & \lambda_{n+3} \end{pmatrix}\begin{pmatrix} -4 & 3 \\ 3 & -1 \end{pmatrix}$$ **1,1**

Selecting the (2, 2) element on each side gives

$$f_n = \tfrac{1}{5}(3\lambda_{n+2} - \lambda_{n+3})$$ **1**

so $s = \frac{3}{5}, t = \frac{1}{5}$ **1**



The Millennium Pyramid is constructed with a hexagonal base and six isosceles triangular faces. The vertices of the base are the points $A(2, 0, 0)$, $B(1, \sqrt{3}, 0)$, $C(-1, \sqrt{3}, 0)$, $D(-2, 0, 0)$, $E(-1, -\sqrt{3}, 0)$ and $F(1, -\sqrt{3}, 0)$. The apex G of the pyramid is the point $(0, 0, 3)$.

(a) Find the equation of the plane containing the triangle ABG. (4)

(b) The angle between two faces is defined to be the acute angle between their normals.

Find (i) the angle between the face ABG and the base; (3)

(ii) the angle between the faces ABG and AFG. (4)

(c) An inward-pointing spotlight is located at the centroid of each triangular face so that its beam is perpendicular to the face. Find the point on the z-axis which is illuminated by the spotlights. (4)

Note The position vector of the centroid of a triangle PQR is given by

$$\tfrac{1}{3}(\mathbf{p} + \mathbf{q} + \mathbf{r}).$$

			level		Content Reference:		
part	marks	Unit	C	A/B	Main	Additional	Source
(a)	4	3.1	4		3.1.7	3.1.4	2000 SY2 Q8
(b)(i)	3	3.1		3	3.1.8		
(b)(ii)	4	3.1		4	3.1.8		
(c)	4	3.1		4	3.1.9	3.1.5	

(a) One method is to find vectors for two of the sides, form the vector product to obtain a normal and then use one corner to finish the equation.

$$\overrightarrow{FA} = \mathbf{i} + \sqrt{3}\mathbf{j}; \quad \overrightarrow{FG} = -\mathbf{i} + \sqrt{3}\mathbf{j} + 3\mathbf{k} \qquad \mathbf{1}$$

$$\overrightarrow{FA} \times \overrightarrow{FG} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sqrt{3} & 0 \\ -1 & \sqrt{3} & 3 \end{vmatrix} = 3\sqrt{3}\mathbf{i} - 3\mathbf{j} + 2\sqrt{3}\mathbf{k} \qquad \mathbf{2}$$

Hence, equation of AFG is of the form $3x - \sqrt{3}y + 2z = k$ and $k = 6$.

$$3x - \sqrt{3}y + 2z = 6 \qquad \mathbf{1}$$

(b) (i) The normal to the base is **k** and to AFG is $3\mathbf{i} - \sqrt{3}\mathbf{j} + 2\mathbf{k}$. **2**

Applying $\mathbf{a.b} = ab\cos\theta$ where θ is the angle between the normals gives

$$1.\sqrt{9 + 3 + 4}\cos\theta = \mathbf{k}.(3\mathbf{i} - \sqrt{3}\mathbf{j} + 2\mathbf{k}) = 2 \qquad \mathbf{1}$$

i.e. $\cos\theta = \frac{1}{2}$ i.e. $\theta = \frac{\pi}{3} (= 60°)$. **1**

(ii) The normal of EFG is

$$\overrightarrow{EF} \times \overrightarrow{EG} = 2\mathbf{i} \times (3\mathbf{i} - \sqrt{3}\mathbf{j} + 2\mathbf{k}) \qquad \mathbf{1}$$

$$= -6\mathbf{j} + 2\sqrt{3}\mathbf{k} \qquad \mathbf{1}$$

As before,

$$4 \times 4\sqrt{3}\cos\phi = (3\mathbf{i} - \sqrt{3}\mathbf{j} + 2\mathbf{k}).(-6\mathbf{j} + 2\sqrt{3}\mathbf{k}) = 10\sqrt{3} \qquad \mathbf{1}$$

$$\Rightarrow \cos\phi = \frac{10\sqrt{3}}{12\sqrt{3}} = \frac{5}{8} \Rightarrow \phi \approx 51{\cdot}32° \qquad \mathbf{1}$$

(c) Centroid of ΔEFG is $\left(0, -\frac{2\sqrt{3}}{3}, 1\right)$ **1**

so the equation of the line of the beam from this point is

$$\frac{x - 0}{0} = \frac{y + \frac{2\sqrt{3}}{3}}{-6} = \frac{z - 1}{2\sqrt{3}} = t \qquad \mathbf{1}$$

This line meets the z-axis when $y = 0$, i.e. when $t = -\frac{\sqrt{3}}{9}$. **1**

Thus $z = 1 - 2\sqrt{3} \times \frac{\sqrt{3}}{9} = 1 - \frac{2}{3} = \frac{1}{3}$. Point is $\left(0, 0, \frac{1}{3}\right)$. **1**