Advanced Higher Maths: Formulae

Green (G): Formulae you absolutely <u>must</u> memorise in order to pass Advanced Higher maths. *Remember you get **no formula sheet at all** in the exam!*
Amber (A): You don't have to memorise these formulae, as it is possible to derive them from scratch in the exam. But… it will save you a lot of time if you do choose to memorise them, and I advise that you do.
Red (R): Don't worry about memorising these. Just use this sheet to help jog your memory in classwork and homework. One or two of these formulae are on the syllabus, but are sufficiently obscure that I don't think it essential to memorise them.

Essential Trigonometric Identities: (from Intermediate 2 and Higher)

	Essential Formulae to know by heart for the exam (G)	**Other useful ones that may be useful for homework/classwork etc.**
Links between ratios	$\cos^2 A + \sin^2 A = 1$ $\tan A = \dfrac{\sin A}{\cos A}$	$1 + \tan^2 A = \sec^2 A$ $\cot^2 A + 1 = \text{cosec}^2 A$ (A)
Compound Angle	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$	$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ (R)
Double Angle	$\sin(2A) = 2\sin A \cos A$ $\cos(2A) = \cos^2 A - \sin^2 A$	$\tan(2A) = \dfrac{2\tan A}{1 - \tan^2 A}$ (R)
Squared	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	

Unit 1.1: Binomial Theorem

The coefficient of the r^{th} term in the binomial expansion $(x+y)^n$ is $\binom{n}{r} x^{n-r} y^r$

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Unit 2.3: Complex Numbers

For the complex number, $z = a + bi$,

- the **modulus** is given by $|z| = \sqrt{a^2 + b^2}$
- and the **argument** is given by $\tan\theta = \dfrac{b}{a}$ $\quad -\pi < \theta < \pi$

De Moivre's Theorem says that

for any $z = r(\cos\theta + i\sin\theta)$, then $z^n = r^n(\cos n\theta + i\sin n\theta)$

Units 1.2 and 2.1: Differentiation

$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\cot x$	$-\text{cosec}^2 x$
$\ln f(x)$	$\frac{f'(x)}{f(x)}$

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

To differentiate an inverse function: $\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$ (A)

Parametric Equations (where $x = f(t), y = g(t)$):

- Gradient (direction of movement) $= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- Speed $= \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}$
- $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$ or $\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}$ (A)

Units 1.3 and 2.2: Integration

(G) Essential Integrals to Learn

$f(x)$	$\int f(x)dx$
$\sec^2 x$	$\tan x + C$
$\tan x$	$\ln\lvert\sec x\rvert + C$
$\frac{f'(x)}{f(x)}$	$\ln\lvert f(x)\rvert + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$

(A) Could use substitution if needed:

$f(x)$	$\int f(x)dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + C$

(R) To save you time in hard questions for homework/classwork, no need to memorise:

$f(x)$	$\int f(x)dx$
$\text{cosec}x$	$-\ln\lvert\text{cosec}\, x + \cot x\rvert + C$
$\cot x$	$\ln\lvert\sin x\rvert + C$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + C$

Volume of solid of revolution *f(x)* about *x* axis: $V = \pi\int_a^b f(x)^2 dx$

Unit 2.4: Sequences and Series

Arithmetic Series	Geometric Series	
$u_n = a + (n-1)d$	$u_n = ar^{n-1}$	
$S_n = \frac{1}{2}n(2a + (n-1)d)$	$S_n = \dfrac{a(1-r^n)}{1-r}$	$r \neq 1$
	$S_\infty = \dfrac{a}{1-r}$	$\lvert r \rvert < 1$

In particular, you are supposed to know that as a consequence of the last formula (A):

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \ldots$$

and $$\frac{1}{a+b} = \tfrac{1}{a}\left(1 - \tfrac{b}{a} + \left(\tfrac{b}{a}\right)^2 - \ldots\right)$$

and also $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$ (also mentioned specifically on syllabus) (R)

Important Identities

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1) \qquad \sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$ (note: this is named specifically on syllabus) (A)

$$\sum_{k=1}^{n} k^3 = \tfrac{1}{4}n^2(n+1)^2$$ (note: this is named specifically on syllabus) (A)

(also note: this is the same as $\left(\sum_{k=1}^{n} k\right)^2$: realising this may help memorise it)

Unit 3.3: Maclaurin Series

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots \quad \text{(G)}$$

and in particular:

Very useful to memorise (A):

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

On syllabus but less essential (R):

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

Unit 3.1: Vectors, Lines and Planes (G)

Angle between two vectors: (Higher) $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$

Equations of a line:

Parametric form	Symmetric/Cartesian form
$x = a + tl$ $y = b + tm \quad (\mathbf{x} = \mathbf{a} + t\mathbf{d})$ $z = c + tn$	$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = t$

Equations of a plane:

Normal **n** is $\begin{pmatrix} l \\ m \\ n \end{pmatrix}$ Point on line = P (with position vector **a**)

Vector equation	Symmetric/Cartesian	Parametric
$\mathbf{x} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n}$ or $(\mathbf{x} - \mathbf{a}) \bullet \mathbf{n} = 0$	$lx + my + nz = k$ where $k = \mathbf{a} \bullet \mathbf{n}$	$\mathbf{x} = \mathbf{a} + \mu\mathbf{b} + \lambda\mathbf{c}$ (**b** and **c** are any two non-parallel vectors in plane)

Angle between two planes = Angle between their normals

Angle between line and plane = (Angle between **n** and **d**) – 90°

Cross product: $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Scalar triple product: $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Unit 3.2: Matrices (G)

		Determinant and Inverse
2×2 matrices	$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$	$\det A = ad-bc$ and $A^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
3×3 matrices	$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$	$\det A = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$

Transformation Matrices

Reflection in x-axis $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, Reflection in y-axis $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Enlargement by scale factor a $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, Rotation by θ degrees $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Unit 3.4: Differential Equations (G)

For $\frac{dy}{dx}+P(x)y=Q(x)$, the Integrating Factor $I(x)$ is $e^{\int P(x)dx}$

and the solution is given by $I(x)y=\int I(x)Q(x)dx$

Second Order Differential Equations

Nature of roots	Form of general solution
Two distinct real m and n	$y = Ae^{mx}+Be^{nx}$
Real and equal m	$y = Ae^{mx}+Bxe^{mx}$
Complex conjugate $m = p \pm iq$	$y = e^{px}(A\cos q + B\sin qx)$