1. Differentiate the following, simplifying your answers where possible.

(a) $y = \sin^{-1}(\cos x)$ (b) $y = \ln x \cos^{-1}(x)$ (c) $f(x) = \tan^{-1}\left(\dfrac{2\sqrt{x}}{1-x}\right)$

2. Find the equation of the tangent to the curve $xy + y^2 - 3x^2 = 2$ at the point (1,-1).

3. Find the Cartesian equation of the curves that are defined parametrically by

(a) $x = 2\sin\theta, y = \cos^2\theta$ (b) $x = t(t-1), y = 1+t$

4. Use logarithmic differentiation to find the derivatives of

a) $f(x) = xe^{-x}\cos x$ b) $y = \dfrac{(2x+3)^{5/2}}{x(x-1)^{2/3}}$

5. A curve is defined by the parametric equations:

$$x = \frac{(1-t)}{(1+t)}, y = (1-t)(1+t)^2$$

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

(b) Find the equation of the tangent to the curve at the point where $t = 2$.

6. If $x^2 - 2y^2 = 2x$ find the value of $\dfrac{d^2y}{dx^2}$ at the point (4,2).

7. If $x = t^2\sin 3t$ and $y = t^2\cos 3t$, find $\dfrac{dy}{dx}$ in terms of t, and show that the curve defined by these parametric equations is parallel to the x-axis at points where $\tan 3t = \dfrac{2}{3t}$.

Advanced Higher Unit 2 **Further Integration Homework Outcome 2**

1. Find the values of the following integrals

a) $\int_3^6 \frac{dx}{\sqrt{36-x^2}}$ b) $\int_0^2 \frac{dx}{25+x^2}$ c) $\int_{\sqrt{3}}^{3\sqrt{3}} \frac{4}{54+6x^2}dx$

2. Carry out the following integrations

a) $\int \frac{2x^2-2x+3}{(2x-1)(x^2+1)}dx$ b) $\int \frac{x^3+x}{(x+1)(x-2)}dx$

(Hint: use partial fractions)

3. Use integration by parts to evaluate the definite integrals:

a) $\int_0^{2\pi} x\sec^2 x dx$

b) $\int_0^{\frac{1}{2}} xe^{-2x}dx$, express your answer in terms of e

4. In a town with population 40 000, a flu virus spread rapidly last winter. The **percentage** P of the population infected t days after the initial outbreak satisfies the differential equation

$$\frac{dP}{dt} = kP, \text{ where } k \text{ is a constant.}$$

(a) If 100 people are infected initially, find in terms of k, the percentage infected t days later.

(b) Given that 500 people have flu after 7 days, how many more are likely to have contracted the virus after 10 days?

5. Use integration by parts to show that

$$\int e^{-3x}\sin 4x dx = -\frac{4}{25}e^{-3x}\left(\cos 4x + \frac{3}{4}\sin 4x\right) + C$$

1. Express the complex number $z = \dfrac{1+3i}{1-2i}$ in the form $x+iy$, where x and y are real numbers. Illustrate your answer on an Argand diagram.

   Determine the modulus and the argument of this complex number.

   Hence express z in polar form.

2. Find the square roots of $7 - 24i$.

3. Verify that $z = 1 + i$ is a root of the equation

   $$z^4 + 3z^2 - 6z + 10$$

   and find the other roots.

4. Find the 6^{th} roots of unity and mark the corresponding points on an Argand diagram.

5. Expand $(\cos\theta + i\sin\theta)^4$ using the binomial theorem and using de Moivre's theorem. Use your expansions to express $\cos 4\theta$ as a polynomial in $\cos\theta$.

6. Evaluate $\left(\dfrac{\sqrt{3}+i}{2}\right)^3$

7. Interpret geometrically in the complex plane the equation $|z+3i| = |z-1|$.

8. (a) If $z = \cos\theta + i\sin\theta$, find in terms of θ

   (i) $z - z^{-1}$

   (ii) $z^n - z^{-n}$

   (b) Hence, using the binomial theorem, express $\sin^5\theta$ in terms of sines of multiples of θ.

Advanced Higher Unit 2 **Sequences and Series Homework Outcome 4**

1. The sum of the first n terms of an arithmetic series is n(n+5).
Find a formula for the nth term.

2. a) In an arithmetic series of 9 terms, the first term is 5 and the last term is 23. Find the sum of the 9 terms.

b) Find the sum of the first 7 terms of a geometric series that has eighth term $\frac{2}{3}$ and fifth term 18.

3. How many terms of the following series must be taken to make the sum exceed 10^5?

a) $3 + \frac{11}{2} + 8 + \ldots$ b) $4 + 6 + 9 + \ldots$

4. Show that the S_{10} of the series $3 + 2 + \frac{4}{3} + \ldots$ differs from the S_∞ by less than $\frac{1}{6}$.

5. Find a) $\sum_{k=11}^{20}(2k+1)$ b) $\sum_{k=1}^{\infty}\frac{1}{2^{k-1}}$

6. Expand the following as geometric series and state the necessary condition on x for each series to be valid.

a) $\frac{1}{4-x}$ b) $\frac{1}{3+x}$

7. Show that the sum to infinity of the series

$$10 + 4\sin^2\theta + \frac{8}{5}\sin^4\theta + \ldots$$

is given by $S_\infty = \frac{50}{4+\cos 2\theta}$

8. The first, second and third terms of an arithmetic series are p, q and p^2 respectively, where $p < 0$.

The first, second and third terms of a geometric series are p, p^2 and q respectively .

a) Show that $p = -\frac{1}{2}$ and find the value of q.

b) Find the sum to infinity of the geometric series.

c) Find the sum of the first 20 terms of the arithmetic series.

Advanced Higher Unit 2 and 3 **Number Theory Homework**

1. Prove by contradiction that $\sqrt{11}$ is irrational.

2. Prove by induction that

$$\sum_{r=1}^{n}(-1)^{r-1}r^2 = \frac{1}{2}(-1)^{n-1}n(n+1) \quad \forall\, n \geq 1 \in N$$

3. Prove that if a and b are odd numbers, then $a + b$ is even.
Use a counter example to show that the converse is false.

4. Show that (3071, 2183) is 37.
Hence find integers x and y such that 37 = x(3071) + y(2183)

5. Show that for $n \geq 0$, where n is an integer, $3^{2n} - 5$ is always a multiple of 4.

6. Prove by induction that

$$\sum_{r=1}^{n}\frac{1}{(3r-2)(3r+1)} = \frac{n}{3n+1} \quad \forall\, n \geq 1 \in N$$

Hence find the sum of the finite series

$$\frac{1}{22\cdot 25} + \frac{1}{25\cdot 28} + \frac{1}{28\cdot 31} + \ldots\ldots + \frac{1}{37\cdot 40}$$

7. Evaluate the sum $\sum_{r=1}^{n}\frac{r}{(r+1)!}$ for $n = 1, 2, 3$ and 4.

Using this, conjecture a formula for the sum and use induction to prove your conjecture.

1. (a) Determine whether $f(x) = x^2 \sin x$ is odd, even or neither. Justify your answer.

(b) Use integration by parts to find $\int x^2 \sin x \, dx$.

(c) Hence find the area bounded by $y = x^2 \sin x$, the lines $x = -\frac{\pi}{4}, x = \frac{\pi}{4}$ and the x-axis.

2. For the curve with parametric equations

$$x = 2\theta + \sin 2\theta, \qquad y = \cos 2\theta$$

Prove that $\frac{dy}{dx} = -\tan\theta$ and $\frac{d^2y}{dx^2} = -\frac{1}{4}\sec^4\theta$.

Hence prove that the curve has a stationary point at (2π, 1) and investigate its nature.

3. The first three terms of a geometric sequence are

$$\frac{x(x+1)}{(x-2)}, \frac{x(x+1)^2}{(x-2)^2} \text{ and } \frac{x(x+1)^3}{(x-2)^3}, \text{ where } x < 2$$

(a) Obtain an expression for the *n*th term of the sequence.

(b) Find an expression for the sum of the first *n* terms of the sequence.

(b) Obtain an expression for the sum to infinity of the sequence.

4. Let $z = \cos\theta + i\sin\theta$

(a) Use the binomial theorem to show that the real part of z^4 is

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

Obtain a similar expression for the imaginary part of z^4 in terms of θ.

(b) Use de Moivre's theorem to write down an expression for z^4 in terms of 4θ.

(c) Use your answers to (a) and (b) to express $\cos 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(d) Hence show that $\cos 4\theta$ can be written in the form $k(\cos^m\theta - \cos^n\theta) + p$ where *k, m, n* and *p* are integers. State the values of *k, m, n* and *p*.

Outcome 1

1 Find the derivative of the function, $f(x)$, defined by

$$f(x) = \tan^{-1}(x^2),\ -1 \le x \le 1$$ **2**

2 Use implicit differentiation to find an expression for $\frac{dy}{dx}$ for the hyperbola

$$4x^2 - y^2 = 16$$ **3**

3 A curve is defined by the parametric equations $x = 5t^2, y = t^3 + 4$

Find $\frac{dy}{dx}$. **2**

Outcome 2

4 Find $\int \frac{x+6}{x(x+2)}dx$ **3**

5 Use the method of integration by parts to evaluate $\int_0^{\pi} x \sin x \, dx$ **4**

6 Find the general solution of the differential equation $\frac{dy}{dx} = x^2 y$ **2**

Outcome 3

7 $z = 1 + i,\ w = 2 + 3i$ *and* $u = 1 - \sqrt{3}i$ are three complex numbers.

(a) Express zw in Cartesian form and plot zw on an Argand diagram **2**

(b) Find the modulus and argument of u and hence write u in polar form. **3**

Outcome 4

8 For the arithmetic sequence 5, 15, 25… find:

(a) The 30^{th} term. **2**

(b) The sum of the first 30 terms. **2**

9 For the geometric sequence 36, 18, 9…

(a) The 10^{th} term. **2**

(b) An expression for the sum of n terms. **2**

Outcome 5

10 For any real numbers a, b, c and d it is conjectured that

$$a > b \text{ and } c > d \Rightarrow ac > bd$$

Disprove this conjecture by providing a counter-example. **2**

11 Let n be a natural number. Prove by contradiction that if n^3 is odd then so also is n. Hint: suppose n is even i.e. $n = 2k$ for $k \in \mathbb{N}$. **3**

Outcome 1

1 Find the derivative of the function, $f(x)$, defined by

$$f(x) = \sin^{-1}(x^3), \ -1 \le x \le 1$$ **2**

2 Use implicit differentiation to find an expression for $\frac{dy}{dx}$ for the ellipse

$$x^2 + 2y^2 = 16$$ **3**

3 A curve is defined by the parametric equations $x = 3t^4, y = t^2 + 1$

Find $\frac{dy}{dx}$. **2**

Outcome 2

4 Find $\int \frac{5x-8}{x(x-4)} dx$ **3**

5 Use the method of integration by parts to evaluate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ **4**

6 Find the general solution of the differential equation $\frac{dy}{dx} = xy^2$ **2**

Outcome 3

7 $z = 4 + i, \ w = 2 + 3i \ and \ u = -1 + \sqrt{3}i$ are three complex numbers.

(a) Express zw in Cartesian form and plot zw on an Argand diagram **2**

(b) Find the modulus and argument of u and hence write u in polar form. **3**

Outcome 4

8 For the arithmetic sequence 41, 37, 33… find:

(a) The 20^{th} term. **2**

(b) The sum of the first 20 terms. **2**

9 For the geometric sequence 27, 9, 3…

(a) The 20^{th} term. **2**

(b) An expression for the sum of n terms. **2**

Outcome 5

10 For any real numbers c and d it is conjectured that

$$|c|+|d| \leq |c+d|$$

Disprove this conjecture by providing a counter-example. **2**

11 Let n be a natural number. Prove by contradiction that if n^3 is even then so also is n. Hint: suppose n is odd i.e. $n = 2k\text{-}1$ for $k \in \mathbb{N}$. **3**