Advanced Higher Unit 3 **Vectors Homework Outcome 1**

1. Find the unit vectors which are perpendicular to both the vectors $\boldsymbol{u} = \boldsymbol{j} + 4\boldsymbol{k}$ and $\boldsymbol{v} = 3\boldsymbol{i} + 2\boldsymbol{j} + 4\boldsymbol{k}$.

2. a) Find an equation for the plane Π_1 which contains the point $A(1, 1, 0)$, $B(3, 1, -1)$ and $C(2, 0, -3)$.

   b) Given that Π_2 is the plane whose equation is $x + 2y + z = 3$, calculate the size of the acute angle between the planes Π_1 and Π_2.

3. The Lines L_1 and L_2 have equations:

$$L_1 : \frac{x-1}{2} = \frac{y+3}{1} = \frac{z}{-1}$$

$$L_2 : \frac{x-4}{1} = \frac{y+3}{2} = \frac{z+3}{1}$$

   a) Prove that they intersect, and find the point of intersection.

   b) Find the acute angle between the lines in radians.

4. a) Find the coordinates of the point A in which the line l with equation $\frac{x+1}{2} = \frac{y-2}{-1} = \frac{z+3}{2}$ meets the plane Π with equation $3x - y + z = 10$.

   b) Hence find the equation in cartesian form for the line through A lying wholly in the plane and perpendicular to the line.

5. A plane contains the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-1}{2}$ and the plane is parallel to the line $\frac{x}{1} = \frac{y+7}{2} = \frac{z}{3}$. Find the equation of the plane.

6. Find the acute angle between the line

$$x = 3t - 1,\ \ y = t + 1,\ \ z = -t + 3 \text{ and the plane } \pi = 2x - 6y - 5z = 2$$

7. Three planes Π_1, Π_2 and Π_3 have equations:

$$\Pi_1: x - 4y - z = 3$$

$$\Pi_2: 2x - 2y + z = 6$$

$$\Pi_3: 3x - 11y - 2z = 10$$

a) Find the acute angle between Π_1 and Π_2.

b) By using Gaussian elimination, show that the three planes intersect at a point Q, and obtain the coordinates of Q.

c) Find an equation for the line L in which Π_1 and Π_2 intersect, and the point R in which L intersects the xy-plane.

1. Let $A = \begin{pmatrix} k & 1-k \\ 0 & k \end{pmatrix}$, where k is a fixed real number.

Find the matrices A^2, A^3 and A^4, expressing the entries of these matrices in terms of k as simply as possible.

Hence write down the matrix A^n, where n is a positive integer.

2. Given that D is an invertible matrix such that $D^2 = 3D - I$, show that,

a) $D^3 = 8D - 3I$ b) $D^{-1} = 3I - D$

3. Find the value or values of x for which

$$\det\begin{pmatrix} 1 & -3 & 2 \\ 4 & x & 5 \\ 2 & -1 & x+1 \end{pmatrix} = 1$$

4. The 3×3 matrices A and X are given by

$$A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \text{ and } X = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}.$$

By considering the matrix product AX, obtain the matrix B for which $AB = I$ where I is the 3×3 identity matrix.

5. The 2×2 matrix A is given by $A = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}$.

Find the matrix A^2 and the values of the real numbers p and q for which $A^2 = pA + qI$, where I is the 2×2 identity matrix.

Hence, or otherwise:

(a) Find the values of the real numbers x and y for which $A^3 = xA + yI$

(b) Obtain the matrix B for which $AB = I$.

6. Write down the 2×2 matrix A representing a reflection in the x-axis and the 2×2 matrix B representing an anti-clockwise rotation of 30° about the origin. Hence show that the image of a point (x, y) under the transformation A followed by the transformation B is $\left(\frac{kx+y}{2}, \frac{x-ky}{2}\right)$, stating the value of k.

7. The 3×3 matrix A is given by $A = \begin{pmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{pmatrix}$.

It is required to find all the values of the real number λ for which the determinant of the matrix $A - \lambda I$ is zero, where I is the 3×3 identity matrix.

(a) Write down the matrix $A - \lambda I$, where λ is a real number.

(b) Show that the required values of λ satisfy the cubic equation

$$\lambda^3 - \lambda^2 - 21\lambda + 45 = 0,$$

and solve this equation to find all the required values of λ.

Advanced Higher Unit 3 Further Sequences and Series Homework Outcome 3

1. Use Maclaurin's Theorem to **derive** the series expansion for $\cos(x)$, giving the first three non-zero terms. Hence, obtain the first three non-zero terms of the series expansions for:

   (a) $\cos(2x)$
   (b) $\cos^2 x$

2. Find the Maclaurin expansion of $f(x) = \sec x$ as far as the term in x^4.

3. Derive the Maclaurin series expansion as far as the term in x^3 for

$$f(x)=\frac{1}{1+x-2x^2}$$

   (Hint: use partial fractions)

4. Write down the power series expansion of e^x and $\sin x$ as far as the terms in x^4 stating the domain of validity in each case.

   Hence find expansions, as far as the term in x^4, of
   (a) $f(x)=e^x \sin x$ (b) $g(x)=e^{2x}\sin 2x$ (c) $h(x)=e^{\sin x}$

5. A recurrence relation is defined by the formula $x_{n+1}=\frac{1}{2}\left(x_n+\frac{11}{x_n}\right)$.

   Given $x_0 = 3$ calculate x_1, x_2 and x_3 correct to 3 significant figures.

   Find the fixed points of this recurrence relation.

6. Find the first four terms of the Maclaurin series for $(2 + x)\ln(2 + x)$.

7. Using the Maclaurin expansion for e^x, find the coefficient of x^5 in the expansion $\frac{1-3x+x^2}{e^x}$

8. Using the Maclaurin expansions for e^x and $\ln(1 + x)$, show that

$$\ln\left(\frac{1+e^x}{2}\right)=\frac{1}{2}x+\frac{1}{8}x^2+\ldots$$

Advanced Higher Unit 3 **Differential Equations Homework Outcome 4**

1. Find the general solution of the following differential equations:

(a) $x\frac{dy}{dx} - 2y = \sqrt{x}$

(b) $\frac{dy}{dx} = y\tan x - 2\sin x$

2. Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0, \text{ given that when } x = 0,\ y = 1 \text{ and } \frac{dy}{dx} = 2$$

3. Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 2e^{-x}, \text{ for which } y = 0 \text{ and } \frac{dy}{dx} = 1 \text{ when } x = 0$$

Hence, show that the maximum value of y is $\frac{1}{4}$.

4. The differential equation

$$L\frac{di}{dt} + Ri = E$$

occurs in electrical theory, L, R and E being positive constants. Given that i=0 when $t = 0$, find i as a function of t.

Hence, show that as t increases indefinitely, i approaches the value $\frac{E}{R}$.

5. Find the solution of the differential equation

$$\frac{d^2x}{dt^2} + 4x = 6\sin t$$

which satisfies the conditions:

$$\left.\begin{aligned} x &= 0 \\ \frac{dx}{dt} &= 0 \end{aligned}\right\} \text{when } t = 0$$

1. Use Maclaurin's Theorem to find a power series for $\tan x$ as far as the term in x^3.

Hence show that the first three non-zero terms in the power series for $\frac{1+\tan x}{e^x}$ are $1-\frac{x^2}{2}+\frac{2x^3}{3}$.

2. Consider each of the following statements about positive integers. If true, give a proof and if false, give a counter example.

(i) $m \mid (a-1) \Rightarrow m \mid (a^5-1)$

(ii) $a \mid c$ and $b \mid c \Rightarrow ab \mid c^2$

(iii) $ab \mid c^2 \Rightarrow a \mid c$ and $b \mid c$

3. Find the solution of the differential equation

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+5y=4e^{-x}$$

for which $y=\frac{dy}{dx}=0$ at $x=0$.

4. Obtain an equation for the plane passing through the point P (1, 1, 0) which is perpendicular to the line L given by

$$\frac{x+1}{2}=\frac{y-2}{1}=\frac{z}{-1}$$

Find the coordinates of the point Q where the plane and L intersect.

Hence, or otherwise, obtain the shortest distance from P to L and explain why this is the shortest distance.

5. (a) Let A = $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and B = $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ where a, b, c and $d \in \mathbb{R}$.

Find AB and evaluate the determinants det A, det B and det AB.

By using the fact that det AB = det A × det B deduce the identity

$$\left(a^2+b^2\right)\left(c^2+d^2\right)=\left(ac-bd\right)^2+\left(ad+bc\right)^2$$

(b) Write each of the prime numbers 17 and 41 as the sum of the squares of two positive integers.

Hence use the result from (a) to express 697 as p^2+q^2 and as r^2+s^2 where p, q, r and s are distinct positive integers.

6. (a) Using integration by parts, prove that

$$\int e^{ax}\sin x\,dx=\frac{e^{ax}}{1+a^2}\left(a\sin x-\cos x\right)$$

(b) Find the general solution of the differential equation

$$\frac{dy}{dx}+2y=1+\sin x$$

Outcome 1

1. Given the vectors $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ and $\mathbf{b} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ calculate $\boldsymbol{a} \times \boldsymbol{b}$. **3**

2. Obtain in parametric form, an equation for the line which passes through the points (1, 3, 2) and (1, 1, 3). **2**

3. Find the equation of the plane which has normal vector $\mathbf{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$ And passes through the point (2, 1, 3). **2**

Outcome 2

4. Given the matrices $A = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 2 \\ 0 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

   and $D = \begin{pmatrix} 1 & 4 & 3 \\ 2 & -1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$

   Find (a) $2A - 3B + C$ (b) AC (c) C^{-1} (d) det D

   1, 1, 2, 2

Outcome 3

5. Find the first two non-zero terms of the Maclaurin series for $f(x) = \ln(1-2x)$ **4**

6. The equation $e^{2(x-1)} - 2 = x$ can be rewritten as $x = 1 + \frac{1}{2}\ln(2+x)$.

   By using the simple iterative formula $x_{n+1} = 1 + \frac{1}{2}\ln(2+x_n)$ with $x_0 = 2$, find an approximation to the root of the equation which lies in the interval $1 < x < 2.5$, giving your answer correct to two decimal places. **3**

Outcome 4

7. Obtain the general solution of the first-order linear differential equation

   $\frac{dy}{dx} + 2xy = 3xe^{-x^2}$, in the form $y = f(x)$. **5**

Outcome 5

8. Use proof by induction to show that for all $n \geq 1$,

$$\sum_{r=1}^{n}(2r+1) = n(n+2)$$ **5**

9. Use the Euclidean algorithm to obtain the greatest common divisor of 1148 and 1312. **3**

Outcome 5

8. Use proof by induction to show that for all $n \geq 1$,

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$ **5**

9. Use the Euclidean algorithm to obtain the greatest common divisor of 885 and 1635. **3**