Higher

Mathematics

UNIT 1 OUTCOME 4

Sequences

Contents

HSN21400

OUTCOME 4

Sequences

1 Introduction to Sequences

A **sequence** is an ordered list of objects (usually numbers).

Usually we are interested in sequences which follow a particular pattern. For example, $1, 2, 3, 4, 5, 6, \ldots$ is a sequence of numbers – the "…" just indicates that the list keeps going forever.

Writing a sequence in this way assumes that you can tell what pattern the numbers are following but this is not always clear, e.g.

$$28,\ 22,\ 19,\ 17\tfrac{1}{2},\ \ldots.$$

For this reason, we prefer to have a formula or rule which explicitly defines the terms of the sequence.

It is common to use subscript numbers to label the terms, e.g.

$$u_1,\ u_2,\ u_3,\ u_4,\ \ldots$$

so that we can use u_n to represent the nth term.

We can then define sequences with a formula for the nth term. For example:

Formula	List of terms
$u_n = n$	$1,\ 2,\ 3,\ 4,\ \ldots$
$u_n = 2n$	$2,\ 4,\ 6,\ 8,\ \ldots$
$u_n = \frac{1}{2}n(n+1)$	$1,\ 3,\ 6,\ 10,\ \ldots$
$u_n = \cos\left(\frac{n\pi}{2}\right)$	$0,\ -1,\ 0,\ 1,\ \ldots$

Notice that if we have a formula for u_n, it is possible to work out *any* term in the sequence. For example, you could easily find u_{1000} for any of the sequences above without having to list all the previous terms.

Recurrence Relations

Another way to define a sequence is with a **recurrence relation**. This is a rule which defines each term of a sequence using previous terms.

For example:

$$u_{n+1} = u_n + 2, \; u_0 = 4$$

says "the first term (u_0) is 4, and each other term is 2 more than the previous one", giving the sequence $4, 6, 8, 10, 12, 14, \ldots$.

Notice that with a recurrence relation, we need to work out all earlier terms in the sequence before we can find a particular term. It would take a long time to find u_{1000}.

Another example is interest on a bank account. If we deposit £100 and get 4% interest per year, the balance at the end of each year will be 104% of what it was at the start of the year.

$$\begin{aligned} u_0 &= 100 \\ u_1 &= 104\% \text{ of } 100 = 1{\cdot}04 \times 100 = 104 \\ u_2 &= 104\% \text{ of } 104 = 1{\cdot}04 \times 104 = 108{\cdot}16 \\ &\vdots \end{aligned}$$

The complete sequence is given by the recurrence relation

$$u_{n+1} = 1{\cdot}04 u_n \text{ with } u_0 = 100,$$

where u_n is the amount in the bank account after n years.

EXAMPLE

The value of an endowment policy increases at the rate of 5% per annum. The initial value is £7000.

(a) Write down a recurrence relation for the policy's value after n years.

(b) Calculate the value of the policy after 4 years.

(a) Let u_n be the value of the policy after n years.

So $u_{n+1} = 1{\cdot}05 u_n$ with $u_0 = 7000$.

(b)
$$\begin{aligned} u_0 &= 7000 \\ u_1 &= 1{\cdot}05 \times 7000 = 7350 \\ u_2 &= 1{\cdot}05 \times 7350 = 7717{\cdot}5 \\ u_3 &= 1{\cdot}05 \times 7717{\cdot}5 = 8103{\cdot}375 \\ u_4 &= 1{\cdot}05 \times 8103{\cdot}375 = 8508{\cdot}54375 \end{aligned}$$

After 4 years, the policy is worth £8508·54.

2 Linear Recurrence Relations

In Higher, we will deal with recurrence relations of the form

$$u_{n+1} = au_n + b$$

where a and b are any real numbers and u_0 is specified. These are called **linear recurrence relations** of order one.

Note

To properly define a sequence using a recurrence relation, we must specify the initial value u_0.

EXAMPLES

1. A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream. To compensate, a further 25 ml dose is given every 8 hours.

   (a) Find a recurrence relation for the amount of drug in his bloodstream.

   (b) Calculate the amount of drug remaining after 24 hours.

(a) Let u_n be the amount of drug in his bloodstream after $8n$ hours.

$u_{n+1} = 0{\cdot}78u_n + 25$ with $u_0 = 156$

(b) $u_0 = 156$

$u_1 = 0{\cdot}78 \times 156 + 25 = 146{\cdot}68$

$u_2 = 0{\cdot}78 \times 146{\cdot}68 + 25 = 139{\cdot}4104$

$u_3 = 0{\cdot}78 \times 139{\cdot}4104 + 25 = 133{\cdot}7401$

After 24 hours, he will have 133.74 ml of drug in his bloodstream.

2. A sequence is defined by the recurrence relation $u_{n+1} = 0{\cdot}6u_n + 4$ with $u_0 = 7$.

   Calculate the value of u_3 and the smallest value of n for which $u_n > 9{\cdot}7$.

$u_0 = 7$

$u_1 = 0{\cdot}6 \times 7 + 4 = 8{\cdot}2$

$u_2 = 0{\cdot}6 \times 8{\cdot}2 + 4 = 8{\cdot}92$

$u_3 = 0{\cdot}6 \times 8{\cdot}92 + 4 = 9{\cdot}352$

The value of u_3 is $9{\cdot}352$.

$u_4 = 9{\cdot}6112$

$u_5 = 9{\cdot}76672$

The smallest value of n for which $u_n > 9{\cdot}7$ is 5.

Using a Calculator

Using the ANS button on the calculator, we can carry out the above calculation more efficiently.

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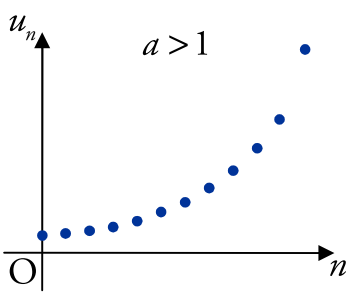
3 Divergence and Convergence

If we plot the graphs of some of the sequences that we have been dealing with, then some similarities will occur.

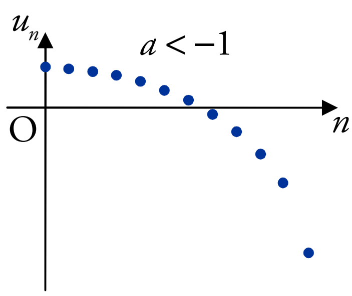
Divergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $a < -1$ or $a > 1$, will have a graph like this:



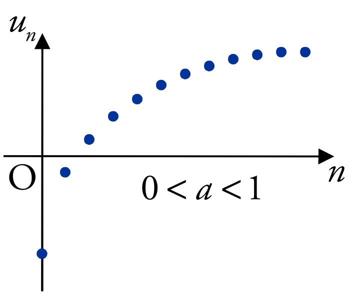
Sequences like this will continue to increase or decrease forever.

They are said to **diverge**.



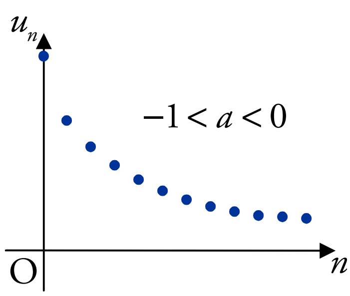
Convergence

Sequences defined by recurrence relations in the form $u_{n+1} = au_n + b$ where $-1 < a < 1$, will have a graph like this:



Sequences like this "tend to a limit".

They are said to **converge**.



4 The Limit of a Sequence

We saw that sequences defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ "tend to a limit". In fact, it is possible to work out this limit just from knowing a and b.

The sequence defined by $u_{n+1} = au_n + b$ with $-1 < a < 1$ tends to a limit l as $n \to \infty$ (i.e. as n gets larger and larger) given by

$$l = \frac{b}{1-a}.$$

You will need to know this formula, as it is not given in the exam.

EXAMPLES

1. The deer population in a forest is estimated to drop by 7·3% each year. Each year, 20 deer are introduced to the forest. The initial deer population is 200.

   (a) How many deer will there be in the forest after 3 years?

   (b) What is the long term effect on the population?

(a)
$$u_{n+1} = 0{\cdot}927u_n + 20$$
$$u_0 = 200$$
$$u_1 = 0{\cdot}927 \times 200 + 20 = 205{\cdot}4$$
$$u_2 = 0{\cdot}927 \times 205{\cdot}4 + 20 = 210{\cdot}4058$$
$$u_3 = 0{\cdot}927 \times 210{\cdot}4058 + 20 = 215{\cdot}0461$$

Therefore there are 215 deer living in the forest after 3 years.

(b) A limit exists, since $-1 < 0{\cdot}927 < 1$.

$$l = \frac{b}{1-a} \qquad \text{where } a = 0{\cdot}927 \text{ and } b = 20$$
$$= \frac{20}{1-0{\cdot}927}$$
$$= 273{\cdot}97 \text{ (to 2 d.p.)}.$$

Therefore the number of deer in the forest will settle around 273.

Note
Whenever you calculate a limit using this method, you must state that "A limit exists since $-1 < a < 1$".

2. A sequence is defined by the recurrence relation $u_{n+1} = ku_n + 2k$ and the first term is u_0.
   Given that the limit of the sequence is 27, find the value of k.

The limit is given by $\dfrac{b}{1-a} = \dfrac{2k}{1-k}$, and so

$$\begin{aligned} \frac{2k}{1-k} &= 27 \\ 27(1-k) &= 2k \\ 29k &= 27 \\ k &= \tfrac{27}{29}. \end{aligned}$$

5 Finding a Recurrence Relation for a Sequence

If we know that a sequence is defined by a linear recurrence relation of the form $u_{n+1} = au_n + b$, and we know three consecutive terms of the sequence, then we can find the values of a and b.

This can be done easily by forming two equations and solving them simultaneously.

EXAMPLE

A sequence is defined by $u_{n+1} = au_n + b$ with $u_1 = 4$, $u_2 = 3{\cdot}6$ and $u_3 = 2{\cdot}04$.

Find the values of a and b.

Form two equations using the given terms of the sequence:

$$\begin{aligned} u_2 &= au_1 + b \\ 3{\cdot}6 &= 4a + b \quad ① \end{aligned} \qquad \text{and} \qquad \begin{aligned} u_3 &= au_2 + b \\ 2{\cdot}04 &= 3{\cdot}6a + b \quad ②. \end{aligned}$$

Eliminate b:

$$\begin{aligned} ①-②:\quad 1{\cdot}56 &= 0{\cdot}4a \\ a &= \frac{1{\cdot}56}{0{\cdot}4} \\ &= 3{\cdot}9. \end{aligned}$$

Put $a = 3{\cdot}9$ into ①:

$$\begin{aligned} 4 \times 3{\cdot}9 + b &= 3{\cdot}6 \\ b &= 3{\cdot}6 - 15{\cdot}6 \\ b &= -12. \end{aligned}$$

So $a = 3{\cdot}9$ and $b = -12$.