National 5 Mathematics
Revision Notes

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Use this booklet to practise working independently like you will have to in the exam.
- Get in the habit of turning to this booklet to refresh your memory.
- If you have forgotten how to do a method, **examples** are given.
- If you have forgotten what a word means, use the **index** (back page) to look it up.

As you get closer to the final test, you should be aiming to use this booklet less and less.

This booklet is for:
- Students doing the National 5 Mathematics course
- Students studying one or more of the National 5 Mathematics units: **Expressions and Formulae**, **Relationships** or **Applications**

This booklet contains:
- The most important facts you need to memorise for National 5 mathematics.
- Examples that take you through the most common **routine** questions in each topic.
- Definitions of the key words you need to know.

Use this booklet:
- To refresh your memory of the method you were taught in class when you are stuck on a homework question or a practice test question.
- To memorise key facts when revising for the exam.

The key to revising for a maths exam is to do questions, not to read notes. **As well as using this booklet, you should also:**
- Revise by working through exercises on topics you need more practice on – such as revision booklets, textbooks, websites, or other exercises suggested by your teacher.
- Work through practice tests
- Ask your teacher when you come across a question you cannot answer
- Use resources online (a link that can be scanned with a SmartPhone is on the last page)

Contents

All information in this revision guide has been prepared in best faith, with thorough reference to the documents provided by the SQA, including the course arrangements, course and unit support notes, exam specification, specimen question paper and unit assessments.

These notes will be updated as and when new information become available. Schools or individuals who purchase a site licence before the first National 5 exam in May 2014 will be eligible to receive a free copy of any revised notes. To enquire further about this, email david@dynamicmaths.co.uk .

We try our hardest to ensure these notes are accurate, but despite our best efforts, mistakes sometimes appear. If you discover any mistakes in these notes, please email us at david@dynamicmaths.co.uk . A corrected replacement copy of the notes will be provided free of charge! We would also like to hear of any suggestions you have for improvement.

This version is version 1.1: published May 2013

Previous versions:
Version 1.0. May 2013.

With grateful thanks to **Arthur McLaughlin** and **John Stobo** for their proof reading.

Formula Sheet

The following formulae are mentioned in these notes and are collected on this page for ease of reference.

Formulae that <u>are given</u> on the formula sheet in the exam (or in unit assessments)

Topic	Formula(e)	Page Reference
The Quadratic Formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	See page 43
Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$	See page 61
Cosine rule	$a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$	See page 62
Area of a Triangle	$A = \frac{1}{2}ab\sin C$	See page 61
Volume of a Sphere	$V = \frac{4}{3}\pi r^3$	See page 25
Volume of a Cone	$V = \frac{1}{3}\pi r^2 h$	See page 25
Volume of a Pyramid	$V = \frac{1}{3}Ah$	See page 24
Standard deviation	$\sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$ or $\sqrt{\frac{\sum x^2 - (\sum x)^2/n}{n-1}}$	See page 79

Formulae that are <u>not</u> given in the exam (or in unit assessments)

Topic	Formula(e)	Page Reference
Pythagoras' Theorem	$a^2 + b^2 = c^2$	See page 46
Right-angled Trigonometry	$\sin x° = \frac{\text{Opp}}{\text{Hyp}}$ $\cos x° = \frac{\text{Adj}}{\text{Hyp}}$ $\tan x° = \frac{\text{Opp}}{\text{Adj}}$	See page 48
Volume of a prism	$V = Ah$	See page 24
Volume of a cylinder	$V = \pi r^2 h$	See page 25
Gradient	$m = \frac{y_2 - y_1}{x_2 - x_1}$	See page 21
Arc length	$\frac{\text{Angle}}{360}\pi d$	See page 23
Sector Area	$\frac{\text{Angle}}{360}\pi r^2$	See page 24
Straight line	$y - b = m(x - a)$	See page 30
Discriminant	$b^2 - 4ac$	See page 44
Trigonometry	$\tan x = \frac{\sin x}{\cos x}$ $\sin^2 x + \cos^2 x = 1$	See page 59
Magnitude of a vector	$\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2}$ $\lvert\mathbf{a}\rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$	See page 70
Semi Interquartile Range (SIQR)	$\frac{\text{upper quartile} - \text{lower quartile}}{2}$	See page 79
Percentage increase and decrease	$\frac{\text{increase (or decrease)}}{\text{original amount}} \times 100$	See page 73

Expressions and Formulae Unit

Surds and Indices

Simplifying Surds

Definition: a **surd** is a square root (or cube root etc.) which does not have an exact answer. e.g. $\sqrt{2} = 1.414213562...$, so $\sqrt{2}$ is a surd. However $\sqrt{9} = 3$ and $\sqrt[3]{64} = 4$, so $\sqrt{9}$ and $\sqrt[3]{64}$ are <u>not</u> surds because they have an exact answer.

We can multiply and divide surds.

Facts

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab} \qquad \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \qquad \sqrt{x} \times \sqrt{x} = \left(\sqrt{x}\right)^2 = x$$

<u>Example 1</u>

Simplify $3\sqrt{2} \times 5\sqrt{2}$

Solution

$$\begin{aligned} 3\sqrt{2} \times 5\sqrt{2} &= 3 \times 5 \times \sqrt{2} \times \sqrt{2} \\ &= 15 \times 2 \qquad \text{(because } \sqrt{2} \times \sqrt{2} = 2\text{)} \\ &= \underline{\underline{30}} \end{aligned}$$

To simplify a surd, you need to look for square numbers that are factors of the original number.

<u>Examples 2</u>

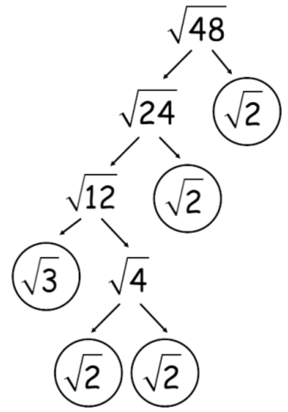
Express $\sqrt{48}$ and $\sqrt{98}$ in their simplest form

Solution

$$\begin{aligned} \sqrt{48} &= \sqrt{16 \times 3} \\ &= \sqrt{16} \times \sqrt{3} \\ &= 4 \times \sqrt{3} \\ &= \underline{4\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \sqrt{98} &= \sqrt{2 \times 49} \\ &= \sqrt{2} \times \sqrt{49} \\ &= \sqrt{2} \times 7 \\ &= \underline{7\sqrt{2}} \end{aligned}$$

Alternative Method (for $\sqrt{48}$): split 48 up into its prime factors – see the diagram on the right (note we could split it up however we wanted and would always end up with the same answer).



From the diagram, we get

$$\begin{aligned} \sqrt{48} &= \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{2} \times \sqrt{3} \\ &= \quad 2 \quad \times \quad 2 \quad \times \sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

You can only add or take away surds when the number underneath the surd sign is the same. e.g. Simplify $\sqrt{5}+\sqrt{3}$ is NOT $\sqrt{8}$. Instead the simplest answer is $\sqrt{5}+\sqrt{3}$ (i.e. no change), because no simplifying is possible.

Examples 3 – simplifying a surd followed by collecting like terms

Write as a single surd in its simplest form: $\boldsymbol{\sqrt{63}+\sqrt{7}-\sqrt{28}}$

Solution

Step One – simplify all three surds

$$\begin{aligned} & \sqrt{63} + \sqrt{7} - \sqrt{28} \\ = & \sqrt{9\times 7} + \sqrt{7} - \sqrt{4\times 7} \\ = & \sqrt{9}\times\sqrt{7} + \sqrt{7} - \sqrt{4}\times\sqrt{7} \\ = & 3\sqrt{7} + \sqrt{7} - 2\sqrt{7} \end{aligned}$$

Step Two – add and take away: $3\sqrt{7}+\sqrt{7}-2\sqrt{7}=\underline{2\sqrt{7}}$

Rationalising the Denominator

For various mathematical reasons, it is not good to have a surd on a bottom of a fraction.

Definition: Rationalising the denominator means turning the surd at the bottom of the fraction into a whole number, whilst keeping the fraction the same.

The method is very simple: **multiply top and bottom by the surd**.

Example 1

Express with a rational denominator: $\dfrac{4}{\sqrt{5}}$

Solution

Multiply top and bottom by $\sqrt{5}$: $\dfrac{4}{\sqrt{5}}\times\dfrac{\sqrt{5}}{\sqrt{5}}=\underline{\dfrac{4\sqrt{5}}{5}}$

Example 2

Express with a rational denominator: $\dfrac{1}{3\sqrt{2}}$

Solution

Multiply top and bottom by $\sqrt{2}$ (not $3\sqrt{2}$): $\dfrac{1}{3\sqrt{2}}\times\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{\sqrt{2}}{3\sqrt{2}\times\sqrt{2}}=\underline{\dfrac{\sqrt{2}}{6}}$

You may be able to simplify your answer after rationalising:

Example 3

Express with a rational denominator: $\dfrac{6}{\sqrt{8}}$

Solution

Multiply top and bottom by $\sqrt{8}$: $\dfrac{6}{\sqrt{8}}\times\dfrac{\sqrt{8}}{\sqrt{8}}=\dfrac{{}^{3}\cancel{6}\sqrt{8}}{{}_{4}\cancel{8}}=\underline{\dfrac{3\sqrt{8}}{4}}$

Rules of Indices

Basic Rule 1: anything to the power 0 is equal to 1:

e.g. $5^0 = 1, \quad 17^0 = 1, \quad 35627658^0 = 1, \quad x^0 = 1$

Basic Rule 2: anything to the power 1 is equal to itself:

e.g. $5^1 = 5, \quad 17^1 = 17, \quad 35627658^1 = 35627658, \quad x^1 = x$

Key Rule 1: when you multiplying two expressions involving powers, you <u>add</u> the numbers in the power: $a^m \times a^n = a^{m+n}$

e.g. $x^3 \times x^4 = \underline{x^7} \qquad y^{-1} \times y^6 = y^{-1+6} = \underline{y^5}$

Key Rule 2: when you divide two expressions involving powers, you <u>take away</u> the numbers in the power: $\dfrac{a^m}{a^n} = a^{m-n}$

e.g. $a^8 \div a^3 = \underline{a^5} \qquad \dfrac{m^{10}}{m^8} = \underline{m^2}$

Key Rule 3: when you take one power to the power of another (nested powers), you <u>multiply</u> the numbers in the power: $(a^m)^n = a^{mn}$

e.g. $(x^2)^3 = \underline{x^6} \qquad (a^4)^{-2} = \underline{a^{-8}}$

<u>Example 1</u>

Simplify $\dfrac{\mathbf{3x^4 \times 8x^8}}{\mathbf{6x^2}}$

Solution

$$\frac{3x^4 \times 8x^8}{6x^2} = \frac{24x^{12}}{6x^2} \quad \text{(add the powers when multiplying)}$$

$$= 4x^{10} \quad \text{(take away the powers when dividing)}$$

<u>Example 2</u>

Simplify $\mathbf{(5x^{-3})^2 \times x^8}$

Solution

$$(5x^{-3})^2 \times x^8 = 25x^{-6} \times x^8 \quad \text{(multiplying the nested powers)}$$

$$= 25x^2 \quad \text{(adding the powers when multiplying)}$$

<u>Example 3</u>

Simplify $\mathbf{3x^2(x^{-2} + 2x^5)}$

Solution

$$3x^2(x^{-2} + 2x^5) = 3x^2 \times x^{-2} + 3x^2 \times 2x^5$$

$$= 3x^{2+(-2)} + 6x^{2+5}$$

$$= 3x^0 + 6x^7$$

$$= \underline{3 + 6x^7}$$

Negative Powers and Fractions in Powers

A **negative power** is to do with dividing. In general, $a^{-m} = \dfrac{1}{a^m}$

e.g. $3^{-2} = \dfrac{1}{3^2} = \underline{\underline{\dfrac{1}{9}}}$ $\quad a^{-4} = \underline{\underline{\dfrac{1}{a^4}}}$ $\quad 5x^{-2} = 5 \times \dfrac{1}{x^2} = \underline{\underline{\dfrac{5}{x^2}}}$

Example 1 – negative power

Rewrite $3x^{-4}$ and $5y^{-7}$ using positive powers

Solution

$\underline{\underline{\dfrac{3}{x^4}}}$ and $\underline{\underline{\dfrac{5}{y^7}}}$

A **fraction as a power** is to do with a root. In general, $a^{\frac{m}{n}} = \sqrt[n]{a}^{\,m}$

e.g. $15^{\frac{2}{3}} = \sqrt[3]{15}^{\,2}$, $\quad a^{\frac{4}{5}} = \sqrt[5]{a}^{\,4}$, $\quad x^{\frac{1}{3}} = \sqrt[3]{x}$

Example 2 – fraction power

Evaluate $9^{\frac{3}{2}}$ and $125^{\frac{4}{3}}$

Solution

$$9^{\frac{3}{2}} = \sqrt[2]{9}^{\,3} = 3^3 = \underline{27}$$

$$125^{\frac{4}{3}} = \sqrt[3]{125}^{\,4} = 5^4 = \underline{625}$$

Hint: calculate the root first and the power second, or the sum may prove too hard for you

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Example 3 – negative and fractional powers

Simplify $25^{-\frac{1}{2}}$

Solution

$25^{-\frac{1}{2}} = \dfrac{1}{25^{\frac{1}{2}}}$ (moving negative power to the bottom)

$= \dfrac{1}{\sqrt{25}}$ (changing fractional power into surd)

$= \underline{\underline{\dfrac{1}{5}}}$ (simplifying the surd)

Scientific Notation (Standard Form)

Definition: Scientific Notation (also known as Standard Form) is a more efficient way of showing really large or really small numbers.

Definition: a number written normally is said to be "in **normal form**".

- A really large number (one that ends with zeroes e.g. 560 000 or 31 million) will have a *positive* number in the power when written in scientific notation
- A really small number (one that begins 'zero point…' e.g. 0.02 or 0.00015) will have a *negative* number in the power when written in scientific notation

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Examples 1

Normal Form	Scientific Notation
92 000 000	9.2×10^{7}
0.000456	4.56×10^{-4}
305 000	3.05×10^{5}

In the exam, you will be expected to use scientific notation in calculations. Make sure you know which buttons to use on your calculator for scientific notation – for instance on some calculators the scientific notation is written $\boxed{\text{EXP}}$ and you do not need to type in the ×10.

Example 2

One light year is approximately $9{\cdot}46 \times 10^{12}$ kilometres.
How many metres are there in 18 light years?
Give your answer in scientific notation.

Solution

First notice that this question asks for the answer to be given in metres, so we have to convert from kilometres to metres. We do this by multiplying by 1000.

Therefore our sum is $9{\cdot}46 \times 10^{12} \times 18 \times 1000$

The answer is $\underline{1{\cdot}7028 \times 10^{17}\text{ metres}}$.

Brackets and Factorising

Single Brackets

At National 4, you learnt to multiply brackets (sometimes referred to as "expanding brackets"). At that level, there was only ever a number in front of a bracket.

Example 1 – numbers in front of brackets

Multiply out the brackets and simplify: $2(3x + 5) + 4(2x - 1)$

Solution

$2(3x+5)+4(2x-1)$

$= 2\times 3x + 2\times 5 + 4\times 2x + 4\times(-1)$

$= 6x + 10 + 8x - 4$ (multiplying out both brackets)

$= \underline{\underline{14x+6}}$ (collecting like terms)

At National 5, there may be letters in front of a bracket. There may also be *both* letters and numbers. The rule remains the same: multiply everything inside the bracket by the number(s) and/or letter(s) outside the bracket.

Examples 2 – letters in front of brackets

Expand the brackets: (the word "expand" makes no difference to the question)

(a) $x(x + 5)$ (b) $2a(3a + 4b)$

Solution

$x(x+5)$	$2a(3a+4b)$
$= x\times x + x\times 5$	$= 2a\times 3a + 2a\times 4b$
$= \underline{x^2 + 5x}$	$= \underline{6a^2 + 8ab}$

Double Brackets

To multiply out double brackets, you have to multiply every term in the first bracket by *every* term in the second bracket. **You always need to simplify your answers – <u>be very careful with negative signs</u>.**

Example 1

Multiply out the brackets: $(x - 7)(x - 9)$

Solution

$(x-7)(x-9)$

$= x\times x + x\times(-9) + (-7)\times x + (-7)\times(-9)$

$= x^2 - 9x - 7x + 63$

$= \underline{x^2 - 16x + 63}$

Example 2

Multiply out the brackets: $(2y + 3)(y - 4)$

Solution

$$\begin{aligned}
&(2y+3)(y-4)\\
&= 2y\times y \;+\; 2y\times(-4) \;+\; 3\times y \;+\; 3\times(-4)\\
&= \quad 2y^2 \qquad -8y \qquad +3y \qquad -12\\
&= \underline{2y^2-5y-12}
\end{aligned}$$

When squaring a bracket, it is important to realise that (for example) $(x+3)^2$ is NOT x^2+9. Instead you have to rewrite $(x+3)^2$ as $(x+3)(x+3)$ and then to multiply out the brackets using the double bracket method.

Example 3 – Squaring brackets

Expand the bracket: $(2a+1)^2$

Solution

$$\begin{aligned}
(2a+1)^2 &= (2a+1)(2a+1)\\
&= 2a\times 2a \;+\; 2a\times 1 \;+\; 1\times 2a \;+\; 1\times 1\\
&= \quad 4a^2 \qquad +2a \qquad +2a \qquad +1\\
&= \underline{4a^2+4a+1}
\end{aligned}$$

In some questions, the second bracket may have three terms in it (a **trinomial**). In these examples, the basic method (multiply everything in the first bracket by everything in the second bracket) is still the same, however you will have to do more multiplications. The answer will often involve a term in x^3 (because $x\times x^2 = x^3$).

Example 4 – three terms in one bracket

Multiply out the bracket: $(x+4)(3x^2-2x+5)$

Solution

$$\begin{aligned}
&(x+4)(3x^2-2x+5)\\
&= x\times 3x^2 \;+ x\times(-2x) \;+ x\times 5 \;+ 4\times 3x^2 \;+ 4\times(-2x) \;+ 4\times 5\\
&= \quad 3x^3 \qquad -2x^2 \qquad +5x \qquad +12x^2 \qquad -8x \qquad +20\\
&= \underline{3x^3+10x^2-3x+20}
\end{aligned}$$

Factorising (Common Factor)

There are three methods you need to be able to use to factorise expressions at National 5:

1. Take a common factor outside of the brackets (the National 4 method)
2. Difference of two squares
3. Trinomials

You should always check your answer by multiplying out the brackets.

Example

Factorise: (a) m^2+7m (b) $2x^2+4xy$

Solution

In (a), the common factor is *m*. The answer is $\underline{\underline{m(m+7)}}$

In (b), *x* is a common factor. However so is 2. Therefore we need to take $2x$ out of the bracket as a common factor. The answer is $\underline{\underline{2x(x+2y)}}$.

Note: the answers $2(x^2+2xy)$ and $x(2x+4y)$ are both technically 'correct', but would only get half marks as they have not been *fully* factorised: you must take *every* possible common factor outside of the bracket.

Difference of two Squares

This method is really basic, but is easily forgotten if you don't practice it regularly. It is called **Difference of Two Squares**. You can spot it because:

- There are only **two** terms
- They are being taken away (a **difference**)
- Both terms are **squares** (letters (x^2, a^2, ...), square numbers (25, 81,...) or both)

The method:
Step 1 – write down two brackets
Step 2 – put a + sign in one bracket and a – sign in the other. (it does not matter which way around they go).
Step 3 – identify what goes at the start and end of each bracket.

Example 1

Factorise a^2-b^2

Solution

Step 1 – write a pair of brackets $(\quad)(\quad)$

Step 2 – write '+' in one bracket and '–' in the other $(\ +\ )(\ -\ )$

(it does not matter which order they go in)

Step 3 – what do we square to make a^2? Write this at the beginning of both brackets.

$(a+\ )(a-\ )$

Step 4 – what do we square to make b^2? Write this at the end of both brackets.

$(a+b)(a-b)$

Final answer: $(a+b)(a-b)$

Example 2

Factorise $4x^2-25$

Solution

Step 1 – write a pair of brackets $(\quad)(\quad)$

Step 2 – write '+' in one bracket and '–' in the other $(\ +\ )(\ -\ )$

(it does not matter which order they go in)

Step 3 – what do we square to make $4x^2$? Write this at the beginning of both brackets.

$(2x+\ )(2x-\ )$

Step 4 – what do we square to make 25? Write this at the end of both brackets.

$(2x+5)(2x-5)$

Final answer: $(2x+5)(2x-5)$

Example 3 – where we must take out a common factor first

Factorise $2x^2 - 32$

Solution

2 and 32 are not square numbers, so we cannot use this method (yet). However we can take out a common factor of 2, to give $2(x^2 - 16)$. We can use the difference of two squares method on $x^2 - 16$, so we can complete the factorising.

Final answer: $2(x+4)(x-4)$

Factorising Trinomials: Basic Method

A **trinomial** is one that contains a squared letter. Examples include $x^2 + 4x + 3$, $b^2 - 4b - 5$ or $3p^2 + 7p - 6$. These factorise into **two brackets**.

Method when there is no number in front of x^2:
To factorise $x^2 + bx + c$, we need one number to go in each bracket. The two numbers that we need will:

- Multiply to make *c*
- Add to make *b*

Always double check your answer by multiplying the brackets back out.

Example 1

Factorise $x^2 - x - 12$

Solution

We need two numbers that multiply to give –12 and add to give –1.

Step 1 – To make x^2, we need $x \times x$, so write *x* and *x* at the start of each bracket.

$$(x \quad)(x \quad)$$

Step 2 – make a list of all possible pairs of numbers that multiply to give –12:

–12 and +1,	–1 and +12,
–6 and +2,	–2 and +6,
–3 and +4	–4 and +3.

Step 3 – look for a pair of numbers that add to make –1.
Out of the six possibilities, we find that –4 and +3 work.

Step 4 – put these numbers into the two brackets and double check they work.
Final answer: $(x-4)(x+3)$

Multiplying these double brackets back out does give the answer $x^2 - x - 12$, which tells us that our answer is correct

Key fact:
To factorise $ax^2 + bx + c$ (when there is a number in front of the x^2), we need one number to go in each bracket. The two numbers that we need:

- WILL multiply to make *c* but they WILL NOT add to make *b*

The key to this method is to experiment and check

Example 2 – where there is a number before x^2

Factorise $3x^2 + 11x + 6$

Solution

Step 1 – make a list of all possible pairs of numbers – **order matters**.

The two numbers in the bracket will multiply to give **+6**. So possibilities are:

+6 and +1, +1 and +6,
+2 and +3 +3 and +2.

(technically we should also consider –6 and –1; –3 and –2 etc. As these also multiply to give +6. However in this question we can ignore negative numbers as there are no negative signs)

Step 2 – To make $3x^2$, we need $3x \times x$, so write $3x$ and x at the start of each bracket.

$$(3x \quad)(x \quad)$$

Step 3 – experiment – try different pairs of numbers in the bracket. Multiply the brackets out to see if you can get $3x^2 + 11x + 6$.

e.g. first you might try $(3x+6)(x+1)$. But this multiplies out to give $3x^2 + 9x + 6$, so this is NOT the answer.

e.g. now you might try switching the '6' and '1' about to get $(3x+1)(x+6)$. But this multiplies out to give $3x^2 + 19x + 6$, so this is NOT the answer.

e.g. next you might try $(3x+2)(x+3)$. This multiplies out to give $3x^2 + 11x + 6$, which is want we want, so this is the answer.

Final Answer: $(3x+2)(x+3)$

Alternative Method for Factorising Trinomials

There is an alternative method for factorising trinomials. The **advantages** of this method over the 'experiment and check' method are that it ought to work on the first attempt, that we don't need to consider the effect of changing the order of the numbers, and that it will work for *any* trinomial.

The **disadvantages** are that the algebra is more complicated, you need to be good at finding *all* factors of larger numbers, requires a good understanding of the common factor method, and some people find this method harder to remember.

The Alternative Method

To factorise $ax^2 + bx + c$:

1. List all the factors of ac (not just c)
2. Look for the pair of factors that add to make b
3. Split bx up using these factors
4. Factorise each half of the expression using a common factor (single bracket) – you will end up with the same bracket twice.
5. Rewrite as two brackets

It is easiest to understand this method by seeing an example.

Example 1 – (same as Example 2 in last section)
Factorise $3x^2 + 11x + 6$

Solution
Step 1 – list all the factors of 18 (we use 18 because it is 3 × 6)
Order does not matter but positive/negative does.

$+18$ and $+1$, $+9$ and $+2$,
$+3$ and $+6$ -3 and -6,
-18 and -1, -9 and -2.

Step two – look for the pair of factors that add to make *b*.
We need a pair of factors that add to make 11.
This means we use +9 and +2

Step three – split *bx* up using these factors

What this means is we split $11x$ up to be $9x + 2x$ (or $2x + 9x$ – it will work either way)

$$3x^2 + 11x + 6 = 3x^2 + 9x + 2x + 6$$

Step four – factorise each half of the expression by taking out a common factor:

$$\begin{aligned} 3x^2 + 11x + 6 &= 3x^2 + 9x + 2x + 6 \\ &= 3x(x+3) + 2(x+3) \end{aligned}$$

Notice how the two brackets are the same $(x+3)$. This will always happen. If it does not, you have made a mistake.

Step five – rewrite as two brackets

The second bracket is the one that is the same $(x+3)$. The first bracket is what is 'left over' when the $(x+3)$ has been taken out.

$$\begin{aligned} &= 3x(x+3) + 2(x+3) \\ &= (3x+2)(x+3) \end{aligned}$$

Final answer: $(3x+2)(x+3)$

[as always, we should still multiply this bracket back out to double check our answer]

Example 2 – (where the 'guess and check' method would be much less efficient)
Factorise $6x^2 - 11x - 10$

Solution
Step 1 – list all the factors of –60 (we use –60 because it is 6 × –10).
Order does not matter but positive/negative does.

–60 and +1,	+60 and –1,	–30 and +2,	+30 and –2,
–20 and +3,	+20 and –3,	–15 and +4,	+15 and –4,
–12 and +5,	+12 and –5,	–10 and +6,	+10 and –6.

Step two – look for the pair of factors that add to make –11.
Examining the list above, the factors we need are –15 and +4.

Step three – split bx up using these factors

Step four – factorise each half of the expression by taking out a common factor

Step five – rewrite as two brackets

$$
\begin{aligned}
&6x^2 - 11x - 10 \\
&= 6x^2 - 15x + 4x - 10 \\
&= 3x(2x-5) + 2(2x-5) \\
&= (3x+2)(2x-5)
\end{aligned}
$$

Final answer: $(3x+2)(2x-5)$

Completing the Square

The process of writing $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$ is called **completing the square**. The completed square form of the equation is useful because from it we can easily determine the maximum or minimum value of a function. It also has uses when we consider the graphs of these functions (see page 37).

To rewrite $y = x^2 + bx + c$ in the form $y = (x+p)^2 + q$, **we use the fact that the number in the bracket (*p*) is half of the coefficient of *x* (*b*).**

Example

Express $x^2 + 8x + 3$ in the form $(x+p)^2 + q$

Solution

We can immediately see that $p = 4$ (half of 8), so $x^2 + 8x + 3 = (x+4)^2 + q$

Now we expand the bracket and compare to the original expression to work out q :

$$
\begin{aligned}
x^2 + 8x + 3 &= (x+4)^2 + q \\
x^2 + 8x + 3 &= (x+4)(x+4) + q \\
x^2 + 8x + 3 &= x^2 + 8x + 16 + q
\end{aligned}
$$

$$
\begin{aligned}
3 &= 16 + q \\
q &= -13
\end{aligned}
$$

Final answer: $x^2 + 8x + 3 = (x+4)^2 - 13$

Fact: this tells us that the minimum value of $x^2 + 8x + 3$ is –13, and that this occurs when $x = -4$. See page 38 for more information about these functions and how their equation relates to their value and their graph.

Algebraic Fractions

Simplifying Algebraic Fractions

You can simplify a fraction when there is a common factor (number *or* letter) on the top *and* on the bottom.

e.g. $$\frac{9xy^2}{18x^2y} = \frac{\cancel{9}^{1}\,\cancel{x}\,y^{\cancel{2}}}{\cancel{18}_{2}\,x^{\cancel{2}}\,\cancel{y}} = \frac{y}{2x}$$

A factor may also be an entire bracket. In this case you can cancel the entire bracket if the entire bracket is identical on the top and the bottom. In National 5 assessments, questions will be of this form.

Example 1

Simplify the fractions (a) $\frac{(a+5)(a-1)}{(a-1)(a+2)}$ (b) $\frac{(x+3)(x-2)}{(x-2)^2}$

Solution

(a) $$\frac{(a+5)(a-1)}{(a-1)(a+2)} = \frac{(a+5)\cancel{(a-1)}}{\cancel{(a-1)}(a+2)} = \frac{(a+5)}{(a+2)}$$

(b) $$\frac{(x+3)(x-2)}{(x-2)^2} = \frac{(x+3)\cancel{(x-2)}}{(x-2)\cancel{(x-2)}} = \frac{(x+3)}{(x-2)}$$

Important: neither of these answers can be cancelled any further as the remaining brackets are different. You cannot cancel the *a* or the *x*, as they are not common factors.

In exam questions, if you are asked to simplify, you will need to factorise first and then cancel brackets.

Example 2

Simplify $\frac{v^2-1}{v-1}$

Solution

v^2-1 is a difference of two squares (see page 12), which factorises to be $(v+1)(v-1)$:

$$\frac{v^2-1}{v-1} = \frac{(v+1)\cancel{(v-1)}}{\cancel{(v-1)}} = \frac{v+1}{1} = v+1$$

Example 3

Write $\frac{2a^2+a-1}{2a^2+5a-3}$ in its simplest form

Solution

Factorising both lines gives:

$$\frac{2a^2+a-1}{2a^2+5a-3}=\frac{(2a-1)(a+1)}{(2a-1)(a+3)}$$

$$=\frac{\cancel{(2a-1)}(a+1)}{\cancel{(2a-1)}(a+3)}=\frac{a+1}{a+3}$$

Multiplying and Dividing Algebraic Fractions

We multiply and divide fractions in the same way as we do for numerical fractions (shown on page 76). Multiplying fractions is a straightforward procedure – you **multiply the tops and multiply the bottoms**.

e.g. $\frac{3}{5}\times\frac{2}{7}=\frac{3\times 2}{5\times 7}=\frac{6}{35}$ $\quad$ $\frac{a}{c}\times\frac{b}{c}=\frac{ab}{c^2}$

It is easiest to cancel <u>before</u> you multiply. You are allowed to cancel *anything* from the top row with *anything* from the bottom row.

<u>Example 1</u>

Write a single fraction in its simplest form: $\boldsymbol{\frac{a^2}{15b}\times\frac{10}{a}}$ $\quad$ $\boldsymbol{a,b\neq 0}$

Solution

Cancelling gives: $\frac{a^{\cancel{2}}}{\cancel{15}^{3}b}\times\frac{\cancel{10}^{2}}{\cancel{a}}=\frac{a}{3b}\times\frac{2}{1}=\frac{2a}{3b}$. $\quad$ **Answer**: $\frac{2a}{3b}$

To divide two fractions, you:

1. flip the second fraction upside down
2. and change the sum to be a multiply sum:

e.g. $\frac{3}{5}\div\frac{2}{7}=\frac{3}{5}\times\frac{7}{2}=\frac{21}{10}$ $\quad$ $\frac{x}{y}\div\frac{a}{x}=\frac{x}{y}\times\frac{x}{a}=\frac{x^2}{ay}$

<u>Example 2</u>

Write as a single fraction in its simplest form: $\boldsymbol{\frac{6b}{ay}\div\frac{3ab}{x}}$, $\quad$ $\boldsymbol{a,x,y\neq 0}$

Solution

Flip the second fraction upside down and multiply: the 'new' sum becomes $\frac{6b}{ay}\times\frac{x}{3ab}$

$$\frac{\cancel{6}^{2}\cancel{b}}{ay}\times\frac{x}{\cancel{3}_{1}a\cancel{b}}$$ (cancelling common factors)

$$=\frac{2}{ay}\times\frac{x}{a}$$

$$=\underline{\underline{\frac{2x}{a^2y}}}$$ (multiplying the tops, and multiplying the bottoms)

Adding and Subtracting Algebraic Fractions

You can only add and subtract fractions when the denominators are the same. When they are not the same, we have to change the fractions into another fraction that *is* the same.

A quick method for doing this, and the one used in these notes, is known as the **'kiss and smile'** method because of the shape formed when you draw lines between the terms you are combining:

$$\frac{a}{b}+\frac{c}{d}$$

$$\frac{a}{b}+\frac{c}{d}=\frac{\qquad}{bd}$$

Step One (the "smile") – Multiply the two bottom numbers together to get the "new" denominator, which will be the same for each fraction.

$$\frac{a}{b}+\frac{c}{d}=\frac{ad\qquad}{\qquad}$$

Step 2a (the first part of the "kiss")

Multiply diagonally the top left and bottom right terms

$$\frac{a}{b}+\frac{c}{d}=\frac{\qquad bc}{\qquad}$$

Step 2b (the final part of the "kiss")

Multiply diagonally the top right and bottom left terms

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$$

Step 3 –

Insert the add (or take away) sign between the two terms on the top row to get the final answer

---

Example 1

Add, giving your answer as a single fraction in its simplest form:

$$\frac{m}{5}+\frac{3}{m}, \qquad m\neq 0$$

Solution

Step one (smile) $\frac{m}{5}+\frac{3}{m}$

Step two (kiss) $\frac{m}{5}+\frac{3}{m}$

$$\frac{m}{5}+\frac{3}{m}=\frac{\qquad}{5m}$$

$$=\frac{m^2+15}{5m}$$

Final answer: $\frac{m^2+15}{5m}$

Taking away works in exactly the same way as adding. The only difference is that the final answer has a take away sign in place of the add sign.

Example 2

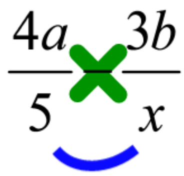
Take away, giving your answer as a single fraction in its simplest form:

$$\frac{4a}{5}-\frac{3b}{x}, \qquad x \neq 0$$

Solution

$$\frac{4a}{5}-\frac{3b}{x}=\frac{\qquad}{5x} \qquad \text{(smile)}$$

$$=\frac{4ax-15b}{5x} \qquad \text{(kiss)}$$



When a fraction has more than one term on the top or bottom (e.g. $x+2$ rather than just x or 2), you need to introduce **brackets** (i.e. $x+2$ becomes $(x+2)$). You then perform kiss and smile, thinking of the bracket as a single object.

Example 3 – brackets

Add, giving your answer as a single fraction in its simplest form:

$$\frac{x}{x+2}+\frac{3}{x-4}, \qquad x \neq -2, x \neq 4$$

Solution

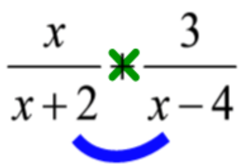
$$\frac{x}{x+2}+\frac{3}{x-4}=\frac{x}{(x+2)}+\frac{3}{(x-4)}$$

$$=\frac{\qquad}{(x+2)(x-4)} \qquad \text{(smile)}$$

$$=\frac{x(x-4)+3(x+2)}{(x+2)(x-4)} \qquad \text{(kiss)}$$

$$=\frac{x^2-4x+3x+6}{(x+2)(x-4)} \qquad \text{(multiplying out brackets on top line)}$$

$$=\frac{x^2-x+6}{(x+2)(x-4)} \qquad \text{(simplifying top line)}$$



We didn't multiply out the bottom line because the bottom line was already in its simplest possible form. However we could have multiplied it out if we chose to.

Gradient

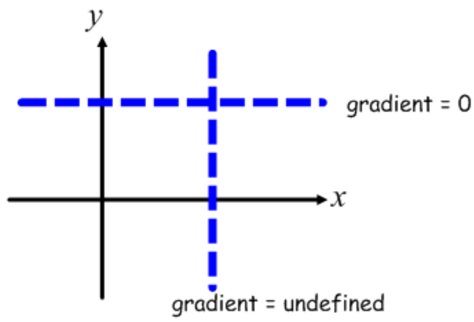
The Meaning of Gradient

The **gradient** of a line is its steepness. It measures how much the line goes up (or down) for every one unit that you move along.

The basic definition of gradient is: $\text{gradient} = \dfrac{\text{up}}{\text{along}} = \dfrac{\text{vertical}}{\text{horizontal}}$.

A **positive** gradient (e.g. 2, $\frac{1}{2}$, $\frac{5}{7}$) means the line slopes **upwards**. A **negative** gradient (e.g. –2, $-\frac{1}{2}$, $\frac{-5}{7}$) means the line slopes **downwards**.

- A gradient of 2 means 'along 1, up 2'.
- A gradient of –3 means 'along 1, down 3'
- A gradient of $\frac{3}{4}$ means 'along 1, up $\frac{3}{4}$', [more easily thought of as 'along 4, up 3']



If a line is horizontal, it has a gradient of **zero**. If a line is vertical, we say its gradient is **undefined**.

We usually use the letter m to mean gradient.

Two lines are **parallel** if they have the same gradient.

Calculating the Gradient

Formula

Gradient between two points (x_1, y_1) and (x_2, y_2): $\boldsymbol{m = \dfrac{y_2 - y_1}{x_2 - x_1}}$

Example 1 – from coordinates

Find the gradient between the points (–2, 5) and (1, 4)

Solution

Step 1 – label the coordinates: $(\underset{x_1}{-2}, \underset{y_1}{5})$ $(\underset{x_2}{1}, \underset{y_2}{4})$

Step 2 – put into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

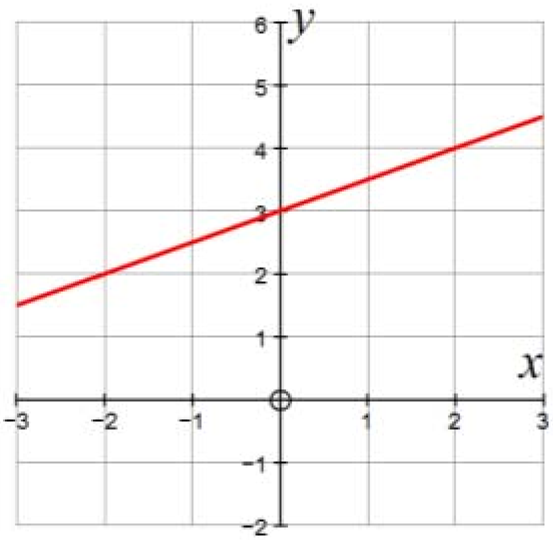
$$m = \frac{4 - 5}{1 - -2}$$

$$= \frac{-1}{3}$$

Answer: $m = -\frac{1}{3}$

Example 2 – from a diagram

Calculate the gradient of this straight line



Solution

Step 1 – identify any two 'nice' coordinates on the line: (0, 3) (2, 4)

Step 2 – label the coordinates:

$(\underset{x_1}{0}, \underset{y_1}{3}) \quad (\underset{x_2}{2}, \underset{y_2}{4})$

Step 3 – put into the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{4-3}{2-0}$$

$$= \frac{1}{2}$$

Answer: $m = \frac{1}{2}$

Perimeter, Area and Volume

Rounding to Significant Figures

Example 1

446.586 → Rounded to 1 significant figure is **400**
→ Rounded to 2 significant figures is **450**
→ Rounded to 3 significant figures is **447**
→ Rounded to 4 significant figures is **446.6**

Example 2

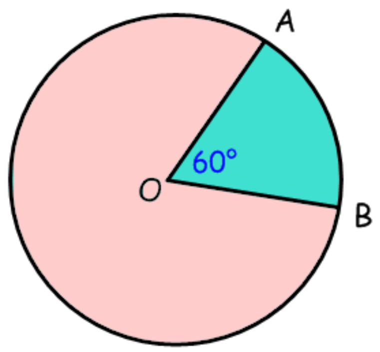
0.00567 → Rounded to 1 significant figure is **0.006**
→ Rounded to 2 significant figures is **0.0057**

Arc Length and Sector Area

An **arc** in a circle is a fraction of its circumference. A **sector** of a circle is a fraction of its area.

If you divide a circle into two bits, you get two sectors - a bigger one (major) and a smaller one (minor). In the diagram on the right:

- The smaller blue sector OAB is the minor sector, with the minor arc AB.
- The larger pink sector OAB is called the major sector, with the longer major arc AB.



The key idea in these questions is to identify the fraction of the circle that is in the question. This depends on the **angle** at the centre of the circle. This fraction is always $\frac{\text{Angle}}{360}$.

Formula. These formula are **not** given on the National 5 exam paper.

Arc length in a circle: Arc length $= \frac{\text{Angle}}{360}\pi d$

Sector area of a circle: Sector Area $= \frac{\text{Angle}}{360}\pi r^2$

You are allowed to use 3.14 instead of π in any calculations. On the non-calculator exam paper, you will *have* to use 3.14

Example 1 – Arc Length

Find the length of the (major) arc RS in this sector of a circle

Solution

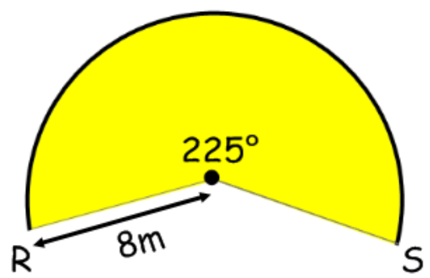
Radius is 8m so diameter is 16m.

Arc length $= \frac{225}{360}\pi d$

$= \pi \times 16 \div 360 \times 225$

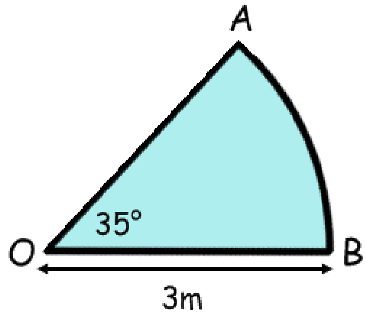
$= 31.41592...$

$= \underline{31.4\text{m (1 d.p.)}}$



Example 2 – Sector area

Calculate the area of (minor) sector AOB in this diagram.



Solution

$$\begin{aligned} A &= \tfrac{35}{360}\pi r^2 \\ &= \pi \times 3^2 \div 360 \times 35 \\ &= 2.74889357... \\ &= \underline{2.75\text{m}^2 \text{ (2 d.p.)}} \end{aligned}$$

Volumes of Solids

You should know from National 4 how to calculate the volume of a **prism**. At National 5 level, you also need to be able to calculate the volume of a **pyramid**.

Formula. This formula is **not** given on the National 5 exam paper.

Volume of a Prism: $V = Ah$

Volume = Area of cross section × Height

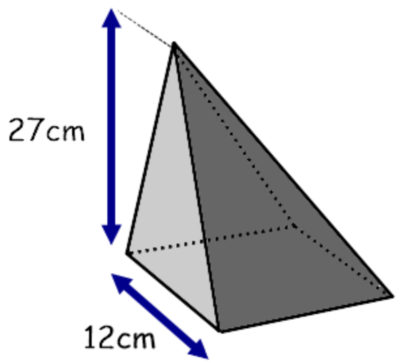
Formula. This formula is given on the National 5 exam paper.

Volume of a Pyramid: $V = \tfrac{1}{3}Ah$

Volume = $\tfrac{1}{3}$ Area of Base × Height

Example 1 – Pyramid

The diagram shows a pyramid with height 27cm and a square base with sides of length 12cm. Calculate the volume of the pyramid.



Solution

The area of a square is given by the formula $A = L^2$, so the area of the base of this pyramid is $12^2 = 144\text{cm}^2$

Therefore the volume of the whole pyramid is

$$\begin{aligned} V &= \tfrac{1}{3}Ah \\ &= 144 \times 27 \div 3 \\ &= \underline{\underline{1296\text{cm}^3}} \end{aligned}$$

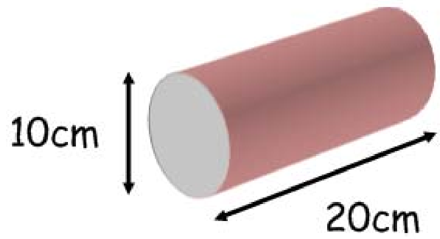
Special cases of prisms and pyramids are when the cross-sectional area of the prism is a circle (in which case you have a cylinder) or when the base of a pyramid is a circle (giving a cone). In these cases, we can adapt the earlier formulae to give us a quicker formula:

Formula. This formula is **not** given on the National 5 exam paper.

Volume of a Cylinder: $V = \pi r^2 h$

Example 2 – cylinder

Work out the volume of this cylinder. Round your answer to 2 significant figures.



Solution

Diameter is 10cm so radius is 5cm

$V = \pi r^2 h$

$= \pi \times 5^2 \times 20$ (or $\pi \times 5 \times 5 \times 20$)

$= 1570.796327....$

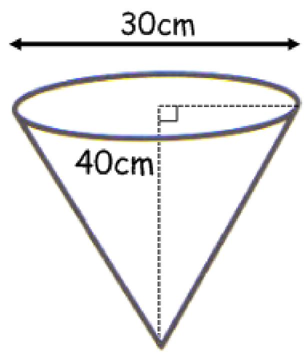
$= \underline{1600\text{cm}^3 \text{ (2 s.f.)}}$

Formula. This formula is <u>given</u> on the National 5 exam paper. **Volume of a Cone:** $V = \frac{1}{3}\pi r^2 h$

In the formula for the volume of a cone, the 'height' h refers to the **perpendicular** height (the one that goes straight up) and <u>not</u> any sloping heights.

Example 3 – cone

Calculate the volume of this cone. Round your answer to 3 significant figures.



Solution

Diameter is 30cm so radius is 15cm

$V = \frac{1}{3}\pi r^2 h$

$= \pi \times 15^2 \times 40 \div 3$ (or $1 \div 3 \times \pi \times 15^2 \times 40$)

$= 9424.777961....$

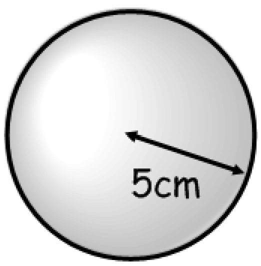
$= \underline{9420\text{cm}^3}$ (3 s.f.)

You are also expected to know how to calculate the volume of a sphere

Formula. This formula is <u>given</u> on the National 5 exam paper. **Volume of a Sphere:** $V = \frac{4}{3}\pi r^3$

Example 4 – sphere

Calculate the volume of this sphere. Round your answer to 1 significant figure.



Solution

Radius is 5cm

$V = \frac{4}{3}\pi r^3$

$= \pi \times 5^3 \div 3 \times 4$ (or $4 \div 3 \times \pi \times 5 \times 5 \times 5$)

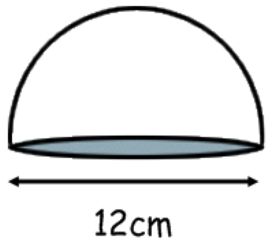
$= 523.5987756....$

$= \underline{500\text{cm}^3 \text{ (1 s.f.)}}$

Definition: A **<u>hemisphere</u>** is half of a sphere.

Example 5 – hemisphere

Calculate the volume of this hemisphere. Round your answer to 4 significant figures.



Solution

Diameter is 12cm so radius is 6cm

$V = \frac{4}{3}\pi r^3 \div 2$

$V = \pi \times 6^3 \div 3 \times 4 \div 2$

$V = 452.3893421....$

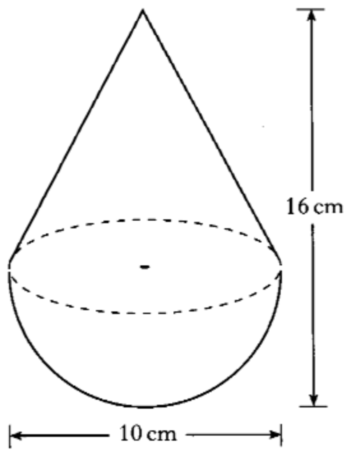
$\underline{V = 452.4\text{cm}^3 \text{ (4 s.f.)}}$

Composite Shapes

In the exam, you may be expected to deal with a shape formed from more than one other shape joined together. If the diagram is confusing you and you are not sure what the shape in the question is, then read the question carefully.

Example (2006 Intermediate 2 Paper 2)

A child's toy is in the shape of a hemisphere with a cone on top, as shown in the diagram. The toy is 10 centimetres wide and 16 centimetres high. Calculate the volume of the toy. Give your answer correct to 2 significant figures.



Solution

The cone and the hemisphere have the same radius, 5cm.

The 16cm line in the picture is made of the height of the cone plus the radius of the sphere.
Height of cone + 5cm = 16cm
i.e. height of cone is 16 – 5 = 11cm.

Volume of cone:

$V = \frac{1}{3}\pi r^2 h$

$= \pi \times 5^2 \times 11 \div 3$

$= 287.98...$

Volume of hemisphere

$V = \frac{4}{3}\pi r^3 \div 2$

$= \pi \times 5^3 \div 3 \times 4 \div 2$

$= 261.8...$

Total volume = 287.98 + 261.8
= 549.7…
= $\underline{550\text{cm}^3}$ (2 s.f.)

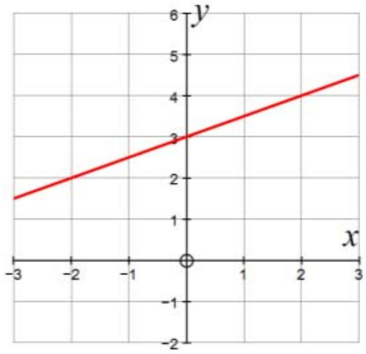
Relationships Unit

Straight Lines

The Equation of a Straight Line

The equation of any straight line is linked to the gradient of the line, and the y-intercept of the line:

- In the expressions and formulae unit, it is explained how to work out the **gradient** of a straight line. For example, it is shown that the gradient of the line on the right is ½ (see page 21).
- **Definition:** the ***y*-intercept** of a straight line is the number on the y-axis that the line passes through. For the line on the right, the y-intercept is 3.



Formula
The equation of any straight line can be written $\boldsymbol{y = mx + c}$, where m is the gradient of the line and c is the y-intercept of the line.

In everyday language, this means that:

- the gradient is "the number before x"
- the y-intercept is "the number that is not before x"

Examples 1

Equation	Gradient	y-intercept
$y = 2x - 5$	2	–5
$y = 8 - x$	–1	8
$y = 4 - 3x$	–3	4
$y = 3 + \frac{5}{2}x$	$\frac{5}{2}$	3

Example 2

What is the equation of the straight line shown at the top of the page?

Solution

The line has gradient ½ and y-intercept 3. Therefore it has equation $\underline{y = \frac{1}{2}x + 3}$

Another method for calculating the equation of a straight line from a diagram or sketch is shown on page 30.

The equation of the line must begin $y = \ldots$ (that is, y has to be the **subject** of the equation). If it does not, the equation must be rearranged so that y is the subject.

Example 3

Find the gradient and y-intercept of the straight lines

(a) $3y = 6x - 9$ (b) $x + y = 5$ (c) $4y - 8x = 4$

Solution

(a) In the equation given, $3y$ is the subject. To make y the subject, we divide through by 3:

$$3y = 6x - 9$$
$$y = 2x - 3$$

Therefore the gradient is 2 and the y-intercept is –3.

(b) In the equation given, $x + y$ is the subject. To make y the subject, we move x to the opposite side:

$$x + y = 5$$
$$y = 5 - x$$

Therefore the gradient is –1 and the y-intercept is 5.

(c) For this equation, we must move the x term across, and divide through by 4. It does not matter which order we do this in.

$$4y - 8x = 4$$
$$4y = 8x + 4 \qquad \text{(moving } -8x \text{ across to become } +8x\text{)}$$
$$y = 2x + 1 \qquad \text{(dividing through by 4)}$$

Therefore the gradient is 2 and the y-intercept is 1.

Drawing a Straight Line from its equation

You need to know how to draw or sketch a line when given its equation. At National 4, you used a table of values to draw a straight line. You can still do this. However there is a quicker way that involves a knowledge of $y = mx + c$.

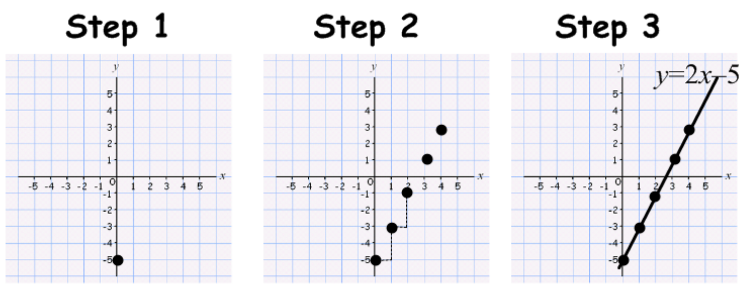
Example 1 – Drawing accurately

Draw the line $y = 2x - 5$

Step 1 – the y-intercept is –5, so the line goes through (0, –5). Plot this point.

Step 2 – the gradient is 2, so move **'along 1 up 2'**. Do this a few times to obtain four or five points.

Step 3 – draw and label the line.



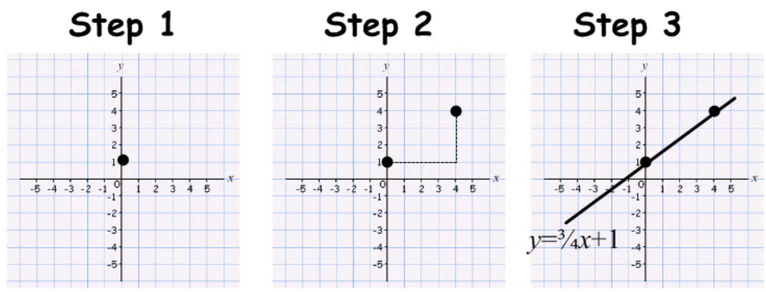
Example 2 – Drawing accurately, fraction gradient

Draw the line $y = \frac{3}{4}x + 1$

Step 1 – the y-intercept is 1, so the line goes through (0,1). Plot this point.

Step 2 – the gradient is $\frac{3}{4}$. This means that you go **'along 4 up 3'** from your first point. Do this a few times to obtain four or five points.

Step 3 – draw and label the line.



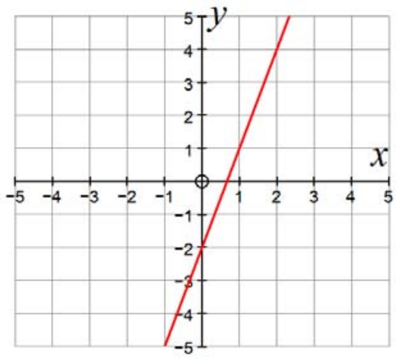
Equation of a graph from a diagram

To find the equation of a straight line from a diagram, you have a choice of two formulae. Which one you use depends on whether or not you are told the y-intercept:

- If you are told the y-intercept, you can use the formula $y = mx + c$.
- If you are not told the y-intercept, there is a second formula $y - b = m(x - a)$

Example 1 – where the y intercept is given

Find the equation of the straight line in this diagram.



Solution

IMPORTANT – show your working to get all the marks. Do not rush straight to step 3.

Step 1 – write down the y-intercept: $c = -2$

Step 2 – calculate the gradient of the line.

Two points on the line are (0,–2) and (1,1). Using the method from page 21:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{1-0} = \frac{3}{1} = 3$$

(you may be able to spot this from the diagram without working. This is OK so long as you clearly write down what the gradient is)

Step 3 – put these values into $y = mx + c$

Answer: $y = 3x - 2$

Formula. This formula **is not** given on the National 5 exam paper.
The equation of the line with gradient m that goes through the point (a , b) is given by:

$$y - b = m(x - a)$$

The only exception is when m is undefined, when the equation of the line is of the form $x = a$

This formula will give the same final answer (once simplified) no matter which point on the line is used.

Example 2 – no *y* intercept given

Find the equation of the line with gradient 4 that goes through the point (5, –1). Rearrange your line into the form $y = mx + c$

Solution

Using $m = 4$ and $(a,b) = (5,-1)$, we have:

$y - b = m(x - a)$

$y - (-1) = 4(x - 5)$

$y + 1 = 4x - 20$ (multiplying out the bracket)

$y = 4x - 20 - 1$ (moving the $+1$ over to the right-hand side)

$y = 4x - 21$ (rearranging into the form $y = mx + c$)

Example 3 – from coordinates

(a) Find the equation of the straight line AB that goes through the points A(1, –1) and B(3, 5) in the form $y = mx + c$

(b) Hence state the *y*-intercept of the line AB

Solution

(a) First, find the gradient.

$m_{AB} = \frac{5 - (-1)}{3 - 1} = \frac{6}{2} = 3$

Then using the formula $y - b = m(x - a)$, with $m = 3$ and the point (3, 5) [this is a random choice, we could just have easily used (1, –1)], we obtain:

$y - b = m(x - a)$

$y - 5 = 3(x - 3)$

$y - 5 = 3x - 9$

$y = 3x - 9 + 5$ (moving -5 over)

$y = 3x - 4$ (simplifying)

(b) Using the answer from part (a), the *y*-intercept is –4.

Equations and Inequations

At National 5 level, you need to be able solve more complex equations and inequations. The method used in these notes is to *change side and do the opposite*. You have to use a method to get the answer – ***if you just write the answer down (even if you think it is obvious) you will get no marks.***

You should always check your final answer so that you know it is correct.

Equations and Inequations with letters on both sides

At National 4, you learnt to solve an equation that has letters on both sides. At National 5 you need to be able to use this method and to extend it to inequations and more difficult equations. The basic method is as follows:

*[**Before you start** (optional) – write the "invisible plus signs" in, in front of anything that does not have a sign in front of it already, to remind you it is positive.]*

- **Step one** – move everything with a letter in to the left-hand side, and all the numbers to the right-hand side, remembering to "change side and do the opposite".
- **Step two** – simplify each side
- **Step three** – solve the resulting equation
- **Final step** – double check your answer works by substituting it back in to both sides of the original equation and checking both sides give the same answer.

Example 1 – equation

Solve algebraically the equation $5y - 5 = 3y + 9$

Solution

Optional first step *– write in "invisible plus signs" in front of anything that does not already have a sign*

Step one – move the '+ 3*y*' over to the left-hand side where it becomes '–3*y*'. Move '–5' to the right-hand side where it becomes '+5'.

Step two – simplify both sides

Step three – divide to get the final answer

$$
\begin{aligned}
5y - 5 &= 3y + 9 \\
+5y - 5 &= +3y + 9 \\
+5y - 3y &= +9 + 5 \\
2y &= 14 \\
y &= \tfrac{14}{2} \\
y &= 7
\end{aligned}
$$

Final step: check, by substituting $y = 7$ into the original equation

- The left-hand side is $5y – 5$. If we replace y with 7, we get $5 \times 7 – 5$, which makes 30.
- The left-hand side is $3y + 9$. If we replace y with 7, we get $3 \times 7 + 9$, which also makes 30. These are the same, so our answer has to be correct.

Inequalities (also known as Inequations) are solved in exactly the same way as for equations, except that there is not a '=' sign in the middle.

Example 2 – inequation

Solve algebraically the inequation $2a + 5 < 15 - 2a$

Solution

Optional first step *– write in "invisible plus signs" in front of anything that does not already have a sign*

Step one – move the '$-2a$' over to the left-hand side where it becomes '$+2a$'. Move '+5' to the right-hand side where it becomes '–5'.

Step two – simplify both sides

Step three – divide to get the final answer. It is fine to leave the final answer as a fraction, so long as it is in its simplest form. (**Note**, we could also have changed to a decimal to give the final answer as $a < 2.5$).

$$\begin{aligned}
2a + 5 &< 15 - 2a \\
+2a + 5 &< +15 - 2a \\
+2a + 2a &< +15 - 5 \\
4a &< 10 \\
a &< \tfrac{10}{4} \\
a &< \tfrac{5}{2} \quad \text{(simplifying)}
\end{aligned}$$

If an equation or an inequation contains a bracket, the bracket can be multiplied out before proceeding with the usual method.

Example 3 – inequation containing brackets

Solve algebraically the inequation $7(x+5) \geq 3x - 2$

Solution

$$\begin{aligned}
7(x+5) &\geq 3x - 2 \\
7x + 35 &\geq 3x - 2 && \text{(multiplying out brackets)} \\
+7x + 35 &\geq +3x - 2 && \text{(optional: writing in invisible + signs)} \\
+7x - 3x &\geq -2 - 35 && \text{(collecting like terms)} \\
4x &\geq -37 && \text{(simplifying)} \\
x &\geq -\tfrac{37}{4}
\end{aligned}$$

Equations and Inequations containing fractions

If an equation or an inequation contains a fraction, we can use a technique called **cross multiplication** to remove the fractions.

Cross multiplication involves multiplying the numerator (top) of each fraction by the denominator (bottom) of the other. For example, $\frac{a}{b} = \frac{c}{d}$ rearranges to become $ad = bc$.

Example 1

Solve the equation $\frac{x}{4} = \frac{1}{x}$

Solution

$$\begin{aligned}
\frac{x}{4} &\times \frac{1}{x} && \text{(cross multiplying)} \\
x \times x &= 1 \times 4 \\
x^2 &= 4 \\
x &= \pm 2
\end{aligned}$$

If either the numerator and/or the denominator of a fraction contains more than just a single letter or number then we must introduce brackets to the expression.

Example 2

Solve the equation $\mathbf{\frac{2}{x+3} = 5}$

Solution

$$\frac{2}{x+3} = 5$$

$$\frac{2}{(x+3)} = 5 \qquad \text{(introducing brackets)}$$

$$2 = 5(x+3) \qquad \text{(cross multiplying)}$$

$$5(x+3) = 2 \qquad \text{(optional: switching sides so that } x \text{ is on the left hand side)}$$

$$5x + 15 = 2 \qquad \text{(multiplying out the bracket)}$$

$$5x = 2 - 15 \qquad \text{(moving } +15 \text{ to the right-hand side)}$$

$$5x = -13$$

$$x = -\tfrac{13}{5}$$

This is not the only method. For example, some people prefer to jump straight to the $5x + 15$ stage without requiring the brackets.

Inequalities (Inequations)

With inequations only, there is one additional rule to bear in mind:

Rule
In an inequation, if you multiply or divide by a *negative* number, the sign reverses (e.g. ≥ becomes ≤ ; < becomes > etc.)

Example

Solve algebraically the inequation $\mathbf{x - 8 > 4x + 7}$

Solution

$$+x - 8 > +4x + 7$$

$$x - 4x > 7 + 8$$

$$-3x > 15$$

$$x < \tfrac{15}{-3} \qquad \text{(changing > to < because we are dividing by a negative)}$$

$$x < -5$$

Simultaneous Equations

Definition: a **simultaneous equation** is where you have more than one equation and more than one unknown letter, and where you have to find values for the letters that fit both equations.

One method that is guaranteed to always work is to multiply to get the same coefficient at the start of each equation and to take them away. However you may need to be careful with negative numbers in this method.

Example 1a: using a method involving taking away

Solve algebraically the system of equations $\begin{aligned}\mathbf{3x-2y=11}\\ \mathbf{2x+5y=1}\end{aligned}$

Solution

Step One – Multiply through each equation by the number at the start of the other equation

e.g. in this example, we multiply the top equation by 2, and the bottom equation by 3

$$\begin{aligned}3x-2y&=11_{\times 2}\\ 2x+5y&=1_{\times 3}\end{aligned}$$

$$\begin{aligned}6x-4y&=22\\ 6x+15y&=3\end{aligned}$$

Step Two – Take away the two equations to eliminate the x terms

$$\begin{array}{lll}\cancel{6x} & -4y & =22\\ \cancel{6x} & +15y & =3\\ \hline & \overset{-4y-15y}{-19y} & \overset{22-3}{=19}\end{array}$$

Step Three – solve the resulting equation: $-19y = 19$, so $y = -1$

Step Four – substitute this value for y back into one of the original equations (either one will do). In this example, we will use the top one:

$$\begin{aligned}3x-2y&=11\\ 3x-2\times -1&=11\\ 3x+2&=11\\ 3x&=9\\ x&=3\end{aligned}$$

Answer: $\underline{x=3,\quad y=-1}$

Step Five – check your answer works by substituting into the second equation.

Alternative strategy: adding. If the equation has both a positive and negative sign in the middle, a better strategy can be to get the same coefficient in the middle and to ***add*** the equations.

Example 1b – same question as Example 1a but using a method involving adding

Solve algebraically the system of equations $\begin{aligned}\mathbf{3x-2y=11}\\ \mathbf{2x+5y=1}\end{aligned}$

Solution

Step One – Multiply through each equation by the number in front of y in the other equation

e.g. in this example, we multiply the top equation by 5, and the bottom equation by 2

$$\begin{aligned}3x-2y&=11_{\times 5}\\ 2x+5y&=1_{\times 2}\end{aligned}$$

$$\begin{aligned}15x-10y&=55\\ 4x+10y&=2\end{aligned}$$

Step Two – Add the two equations to eliminate the y terms

$$\begin{array}{lll}15x & \cancel{-10y} & =55\\ 4x & \cancel{+10y} & =2\\ \hline \overset{15x+4x}{19x} & & \overset{55+2}{=57}\end{array}$$

Step Three – solve the resulting equation: $19x = 57$, so $x = 3$

Step Four – substitute this value for *x* back into one of the original equations (either one will do). In this example, we will use the top one:

$$\begin{aligned} 3x-2y&=11 \\ 3\times 3-2y&=11 \\ 9-2y&=11 \\ -2y&=11-9 \\ -2y&=2 \\ y&=-1 \end{aligned}$$

Answer: $\underline{x=3,\quad y=-1}$

Step Five – check your answer works by substituting into the other equation.

You will also be expected to write your own equations from a real-life situation.

Example 2 – a real life situation

3 pencils and 2 books cost £10.30.
2 pencils and 3 books cost £15.20.

a) Write down a pair of equations to represent this situation

b) Solve these equations algebraically to find the cost of one book and one pencil.

Solution

a) The equations are $\begin{aligned} 3p+2b&=10.30 \\ 2p+3b&=15.20 \end{aligned}$

b) Following the same steps as in the last example:

$$\begin{aligned} 3p+2b&=10.30_{\times 2} \\ 2p+3b&=15.20_{\times 3} \end{aligned}$$

$$\begin{aligned} 6p+4b&=20.60 \\ 6p+9b&=45.60 \end{aligned}$$

$$\begin{array}{rrrl} & \cancel{6p} & +4b & =20.60 \\ - & \cancel{6p} & +9b & =45.60 \\ \hline & & 5b & =25 \\ & & b & =5 \end{array}$$

Substituting back into top equation:

$$\begin{aligned} 3p+2\times 5&=10.30 \\ 3p+10&=10.30 \\ 3p&=0.30 \\ p&=0.10 \end{aligned}$$

Answer: $b = 5$ and $p = 0.10$ (remember to double check)

However you have to answer the question in a sentence to get the final communication mark – i.e. <u>a book is £5 and a pencil is 10p</u>.

Solving simultaneous equations using a graph

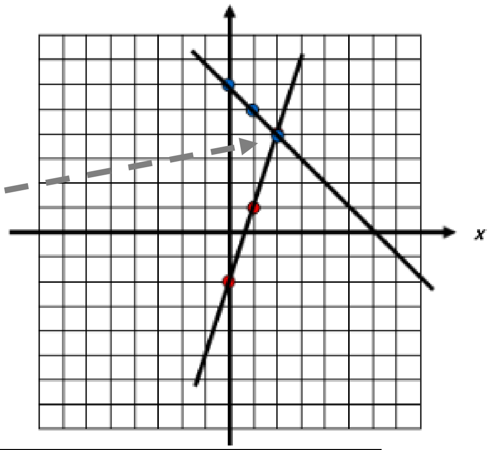
This method is easier, but for various reasons is less likely to come up in the exam.
The solution to two simultaneous equations is the point where the graphs of each equation cross each other. To find this, you need to be able to draw the graphs.

Example

Solve graphically the simultaneous equations $\boldsymbol{y = 3x - 2}$, $\boldsymbol{y = 6 - x}$

Solution

Step 1 – draw the two lines $y = 3x - 2$ and $y = 6 - x$ using the method on page 27.



Step 2 – write down the point where the two graphs cross (the point of intersection)

Answer: the graphs cross each other at the point (2, 4), so the answer is $x = 2$, $y = 4$

Changing the subject of a formula

Changing the subject of a formula was introduced at National 4 and is just like rearranging an equation: you move things from one side to the other and 'do the opposite'.

A useful (but optional) tip for changing the subject questions is to switch the left-hand side and the right-hand side before you begin moving things, so that the letter that will be the subject is already on the left-hand side. However this does not work on every occasion (see example 4)

It is important to move things over in the correct order. Because we are rearranging a formula (effectively going backwards), the order that we deal with terms in is the exact opposite of BODMAS – i.e. the first thing we deal with is terms that are added or subtracted; then we deal with multiplying and division; then squares and square roots, and then lastly split up any brackets.

Example 1

Change the subject of the formula $\boldsymbol{y = ab + d}$ **to '*b*'**

Solution

Step One – flip the left and right hand sides: $ab + d = y$

Step Two – rearrange; dealing with the adding/subtracting first, then multiplying dividing

$$ab + d = y$$

$$ab = y - d \qquad (+d \text{ moves over and becomes } -d)$$

$$b = \frac{y-d}{a} \qquad (\times a \text{ moves over and becomes } \div a)$$

Example 2

Change the subject of the formula $A = \pi d^2$ to 'd'

Solution

Step One – flip the left and right hand sides: $\pi d^2 = A$

Step Two – rearrange; dealing with the multiplying/dividing first, then the square/square root

$$\pi d^2 = A$$
$$d^2 = \frac{A}{\pi} \qquad (\times\pi \text{ moves over and becomes } \div\pi)$$
$$d = \sqrt{\frac{A}{\pi}} \qquad \text{(squaring moves over and becomes square root)}$$

Just because a formula has a square (or square root) in it does not mean that the final answer will have to have a square root (or square) in it. It depends on which letter we are making the subject. If that letter is being squared, then we have to use a square root. If another letter is being squared, then we do not.

Example 3

Change the subject of the formula $P = qt^2 + d$ to 'q'

Solution

Step One – flip the left and right hand sides: $qt^2 + d = P$

Step Two – rearrange; dealing with the adding/taking away, then the multiplying/dividing. There is no need to 'deal with' the square, as q is not being squared

.

$$qt^2 + d = P$$
$$qt^2 = P - d \qquad (+d \text{ moves over and becomes } -d)$$
$$q = \frac{P-d}{t^2} \qquad (\times t^2 \text{ moves over and becomes } \div t^2)$$

There are some occasions in which it does not help to flip the left and right-hand sides. This usually involves formulae where the letter that is to become the subject is on the denominator of the right-hand side.

Example 4

Change the subject of the formula $D = \frac{V}{T}$ to 'T'

Solution

On this occasion, step one is not required. This is because T will move naturally over to the left-hand side when we begin rearranging.

$$D = \frac{V}{T}$$
$$DT = V \qquad (\div T \text{ moves over and becomes } \times T)$$
$$T = \frac{V}{D} \qquad (\times D \text{ moves over and becomes } \div D)$$

Quadratic Functions

Definition: a quadratic function is one that contains a squared term and no higher powers.
e.g. $3x^2$, $x^2 - 4$ and $x^2 + 5x + 1$ are all quadratic functions, but x^5 and $x^2 + x^3$ are not.

Function Notation

You need to be familiar with function notation.

Example

Three functions are defined by $f(x) = x^2 + 4$, $g(x) = 5 - x$ and $h(t) = 4 + 7t$.
Calculate:
(a) $f(7)$ (b) $h(3)$ (c) $g(-2)$

Solutions

$$f(x) = x^2 + 4$$
$$f(7) = 7^2 + 4$$
$$= 53$$

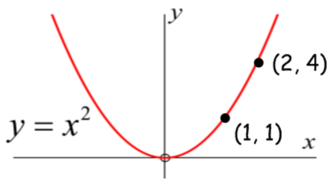
$$h(t) = 4 + 7t$$
$$h(3) = 4 + 7 \times 3$$
$$= 25$$

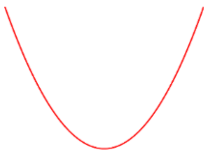
$$g(x) = 5 - x$$
$$g(-2) = 5 - (-2)$$
$$= 7$$

Graphs of Quadratic Functions

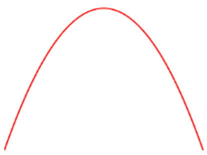
Definition: the graph of a quadratic function is called a **parabola**.

The graph on the right is the basic graph of $y = x^2$. You should know its shape.





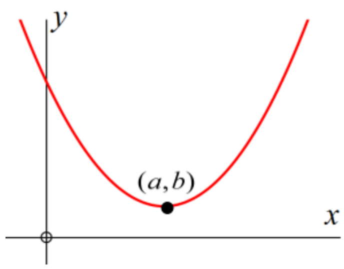
If x^2 is positive, the graph is "happy" (it has a **minimum** turning point)



If x^2 is negative, then the graph is "unhappy" (it has a **maximum** turning point)

The graph of $y = (x - a)^2 + b$ is still a parabola, but it has been moved so that its minimum point is no longer at (0, 0):

- The number inside the bracket (a) tells us how far the graph has been moved **left or right**. If a is positive, the graph moved to the left. If it is negative, it moved to the right.
- The number outside the bracket (b) tells us how far the graph has been moved **up or down**. If b is positive, the graph moved upwards. If it is negative, it moved downwards.



Example A: The graph of $y = (x + 3)^2 - 2$ has been moved 3 to the left and 2 down. Its minimum point is (–3, –2)

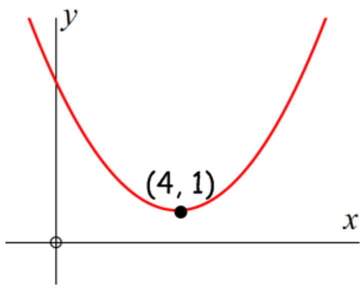
Example B: The graph of $y = (x - 5)^2 + 4$ is a 'happy' parabola that has been moved 5 to the right and 4 up. Its minimum point is (5, 4)

Key facts:

- $y=(x-a)^2+b$ is a "happy" parabola. Its minimum turning point is (a, b)
- $y=-(x-a)^2+b$ is an "unhappy" parabola. Its maximum turning point is (a,b)
- The **axis of symmetry** of $y=(x-a)^2+b$ or $y=-(x-a)^2+b$ has the equation $x=a$

Example 1

This graph has an equation of the form $y=(x-a)^2+b$. What is its equation?



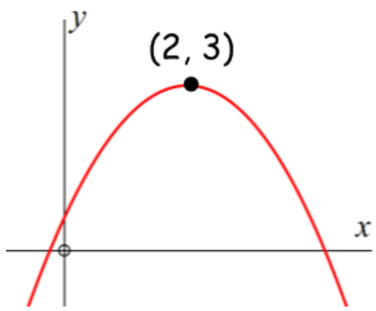
Solution

The minimum point is (4, 1). This tells us that the graph has been moved 4 to the right (so a = 4) and 1 up (so b= 1)

Therefore its equation is $\underline{y=(x-4)^2+1}$

Example 2

What is the equation of this parabola?



Solution

The graph is unhappy, so it has an equation of the form $y=-(x-a)^2+b$ (as opposed to $y=(x-a)^2+b$)

The maximum point is (2, 3). The graph has moved 2 to the right and 3 up. Therefore its equation is $\underline{y=-(x-2)^2+3}$.

Equation of a Graph from a Point on the Graph

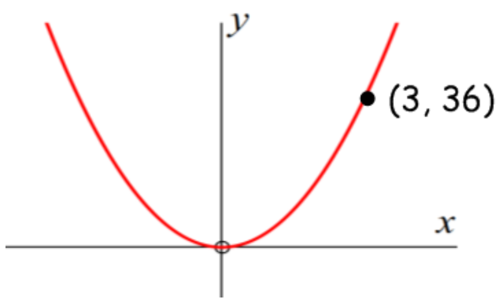
The coordinates of any point on a graph tells you a value for x and y.

e.g. for the coordinate point (3 , 7), $x=3$ and $y=7$
e.g. for the coordinate point (0 , 5), $x=0$ and $y=5$
e.g. for the coordinate point (–4 , 1), $x=-4$ and $y=1$

These values can be put back into the equation of the graph. If you don't know the full equation of a graph, they can give you an equation to solve to complete it.

Example

The graph on the right has the equation $y=kx^2$. The graph passes through the point (3, 36). Find the value of k.



Solution

A point on the graph is (3 , 36). This means that $x=3$ and $y=36$.

Substituting these values into the equation gives:

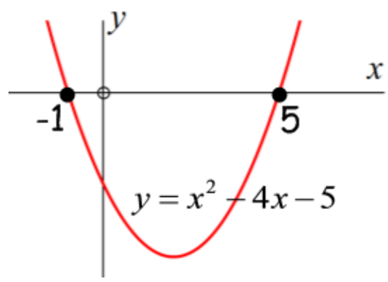
$$y=kx^2$$
$$36=k\times 3^2$$
$$36=k\times 9$$
$$k=4$$

Roots of Quadratic Equations

Definition: the **roots** of a quadratic equation are another word for its solutions.
The roots of a graph of an equation are the points that the graph crosses the x-axis.

Example 1 – from a graph

Using the graph shown, write down the two solutions of the equation $x^2 - 4x - 5 = 0$



Solution

The roots are $x = -1$ and $x = 5$.

Factorising is the simplest way of solving a quadratic equation, but you can only use it when the expression can actually be factorised! See page 11 for help on factorising.

Important – you must rearrange the equation so that is has '= 0' on the right-hand side. If you do not do this, you will risk losing all of the marks.

Example 2 - factorising

Use factorising to solve the equation $2x^2 - 6x = 0$

Solution

Step 1 – check that the equation has '= 0' on the right-hand side.
On this occasion, it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$2x^2 - 6x = 0$$
$$2x(x-3) = 0$$

Step 3 – split up into two separate equations and solve

$$2x = 0 \qquad x - 3 = 0$$
$$x = 0, \qquad x = 3$$

Example 3 - Difference of Two Squares

Use factorising to solve the equation $y^2 - 49 = 0$

Solution

Step 1 – check that the equation has '= 0' on the right-hand side.
On this occasion, it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$y^2 - 49 = 0$$
$$(y+7)(y-7) = 0$$

Step 3 – split up into two separate equations and solve

$$y + 7 = 0 \qquad y - 7 = 0$$
$$y = -7, \qquad y = 7$$

Example 4 – factorising with a coefficient of x^2

Use factorising to solve the equation $2x^2 + 9x - 5 = 0$

Solution

Step 1 – check that the equation has '= 0' on the right-hand side.
On this occasion it does, so we do not need to do anything more.

Step 2 – factorise the expression

$$2x^2 + 9x - 5 = 0$$
$$(2x-1)(x+5) = 0$$

Step 3 – split up into two separate equations and solve

$$2x - 1 = 0 \qquad x + 5 = 0$$
$$2x = 1$$
$$x = \tfrac{1}{2}, \qquad x = -5$$

Example 5 – right-hand side is not equal to zero)

Use factorising to solve the equation $x^2 - 2x - 10 = 5$

Solution

Step 1 – check that the equation has '= 0' on the right-hand side.
It **does not**, so we need to rearrange by moving the 5 over to the left hand side.

$$x^2 - 2x - 10 = 5$$
$$x^2 - 2x - 10 - 5 = 0$$
$$x^2 - 2x - 15 = 0$$

Step 2 – factorise the rearranged expression

$$x^2 - 2x - 15 = 0$$
$$(x+3)(x-5) = 0$$

Step 3 – split up into two separate equations and solve

$$x + 3 = 0 \qquad x - 5 = 0$$
$$x = -3, \qquad x = 5$$

Sketching Parabolas

When making a sketch of a parabola we need to indicate the coordinates of:

- the **roots**. We would usually find the roots by factorising the equation.
- the ***y* intercept**. We can always find this by substituting $x = 0$ into the equation.
- the **turning point**. We can find this by symmetry (using the fact that the turning point is exactly in the middle of the roots). We could also do it by completing the square (see page 16).

Example 1 – factorised

Sketch the graph with equation $y = (x-1)(x+3)$. Mark clearly where the graph crosses the axes and the coordinates of the turning point.

Solution

Shape – the x^2 term is positive, so the graph is a 'happy' parabola (if it was negative, it would be 'unhappy').

Roots – to find the roots, solve the equation $(x-1)(x+3) = 0$:

The bracket $(x-1)$ tells us that one root is $x = 1$.
The bracket $(x+3)$ tells us that the other root is $x = -3$.

Answer: the roots are $x = 1$ and $x = -3$.

***y*-intercept** – to find the *y*-intercept, substitute $x = 0$ into the equation.

$$\begin{aligned} y &= (x-1)(x+3) \\ &= (0-1)(0+3) \\ &= (-1)\times(3) \\ &= -3 \end{aligned}$$

Answer: the *y*-intercept is at $y = -3$.

Turning Point – the simplest way to find the turning point is to use symmetry.

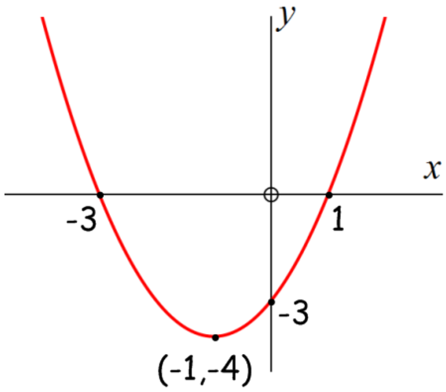
Because the parabola is symmetrical, the turning point is exactly midway between the roots. Since the roots are $x = 1$ and $x = -3$, the turning point must be at $x = -1$.

We now substitute $x = -1$ into the original equation.

$$\begin{aligned} y &= (x-1)(x+3) \\ &= (-1-1)(-1+3) \\ &= (-2)\times(2) \\ &= -4 \end{aligned}$$

Answer: the turning point is (–1, –4).

Sketch – sketch the graph, including all the key points found above. A possible sketch is shown on the right.



Example 2 – completed square

Sketch the graph with equation $y = (x-2)^2 - 9$, showing the coordinates of the turning point and the point of intersection with the *y*-axis.

Solution

Shape – the x^2 term is positive, so the graph is a 'happy' parabola (if it was negative, it would be 'unhappy').

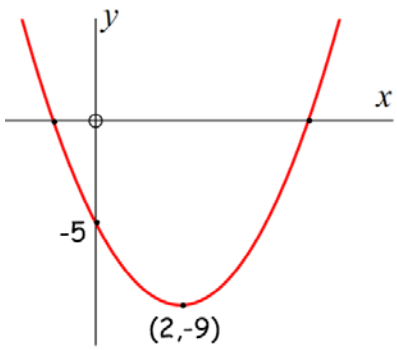
Turning Point – this can be found directly from the equation using the technique on page 38. This tells us that the turning point of $y = (x-2)^2 - 9$ is (2, –9).

***y*-intercept** – to find the *y* intercept, substitute $x = 0$ into the equation.

$$\begin{aligned} y &= (x-2)^2 - 9 \\ &= (0-2)^2 - 9 \\ &= (-2)^2 - 9 \\ &= 4 - 9 \\ &= -5 \end{aligned}$$

Answer: the *y* intercept is at $y = -5$.

Sketch – sketch the graph, including all the key points found above. A possible sketch is shown on the right.



The Quadratic Formula

Formula. This formula is given on the National 5 exam paper.

The roots of $ax^2 + bx + c = 0$ are given by: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula can be used to solve *any* quadratic equation. We usually use it when you can't factorise the expression.

Important – you must rearrange the equation so that is has = 0 on the right-hand side. If you do not do this, you will risk losing all of the marks.

In the National 5 exam, a clue to use the formula (rather than factorising) is where the question tells you to "**give your answers correct to 2** (or 1) **decimal places**" etc.

Example 1

Solve the equation $3x^2 + 2x - 6 = 0$, giving your answers correct to two decimal places.

Solution

Step 1 – check the equation has '= 0' on the RHS. It does, so we can proceed.

Step 2 – write down what a, b and c are: $a = 3$, $b = 2$, $c = -6$

Step 3 – substitute into the formula and solve – **being very careful when dealing with negative signs**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{2^2 - 4 \times 3 \times (-6)}}{2 \times 3}$$

$$x = \frac{-2 \pm \sqrt{4 - (-72)}}{6}$$

$$x = \frac{-2 \pm \sqrt{76}}{6}$$

$$x = \frac{(-2 + \sqrt{76})}{6} \qquad x = \frac{(-2 - \sqrt{76})}{6}$$

$$x = 1.12 \text{ (2 d.p.)} \qquad x = -1.79 \text{ (2 d.p.)}$$

Essential to remember brackets when putting into the calculator!

If the number under the square root sign works out to be negative, then you will not be able to complete the formula. This means either that:

- You have made a mistake with negative numbers and need to check your working (realistically this is the most likely thing that would have happened in an exam)
- Or the equation has no solution (happens a lot in real life, but less likely in an exam)

Example 2

Solve the equation $2x^2 - 5x - 1 = 3$, giving your answers correct to 2 d.p.

Solution

Step 1 – check the equation has '= 0' on the right-hand side. It does not, so we have to rearrange:

$$2x^2 - 5x - 1 - 3 = 0$$
$$2x^2 - 5x - 4 = 0$$

Step 2 – write down what a, b and c are: $a = 2$, $b = -5$, $c = -4$

Step 3 – substitute into the formula and solve – **being very careful when dealing with negative signs**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4 \times 2 \times (-4)}}{2 \times 2}$$
$$x = \frac{5 \pm \sqrt{25 - (-32)}}{4}$$
$$x = \frac{5 \pm \sqrt{57}}{4}$$

Essential to remember brackets when putting into the calculator!

$$x = \frac{(5 + \sqrt{57})}{4} \qquad x = \frac{(5 - \sqrt{57})}{4}$$
$$x = 3.14 \text{ (2 d.p.)} \qquad x = -0.64 \text{ (2 d.p.)}$$

Answer: $x = 3.14$ and $x = -0.64$

The Discriminant

To solve any quadratic equation, we have to use the formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Since this formula contains $\sqrt{b^2 - 4ac}$, and we can only take the square root of a positive number, the value of $b^2 - 4ac$ is important, and has a special name, the **discriminant**. The symbol Δ can be used for the discriminant.

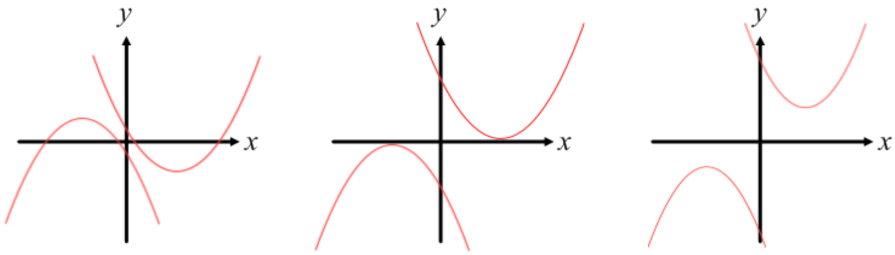
Formula. This formula is **not** given on the National 5 exam paper, though the quadratic formula that contains it is given.
The **discriminant** of a quadratic equation $ax^2 + bx + c = 0$ is given by $\Delta = b^2 - 4ac$

The **discriminant** of the quadratic equation $ax^2 + bx + c = 0$ is the number that has to be square rooted to complete the quadratic formula.

A quadratic equation may have two roots (known as **real and distinct** roots), one root (known as a **real and equal** root) or **no real roots**.
Simply put, the discriminant tells us how many real roots there are to a quadratic equation

- If the discriminant is positive ($\Delta > 0$), the equation has two **distinct real** roots (also known as **unequal real** roots).

- If the discriminant is zero ($\Delta = 0$), the equation has one **real and equal** (or **real and repeated**) root
- If the discriminant is negative ($\Delta < 0$), the equation has **no real roots**.



Two real and distinct roots | One real and equal root | No real roots

---

Example 1

State the nature of the roots of the quadratic equation $2x^2 - x + 5 = 0$

Solution

For this equation $a = 2$, $b = -1$ and $c = 5$

$$\begin{aligned}\Delta &= (-1)^2 - 4\times 2\times 5\\ &= 1 - 40\\ &= -39\end{aligned}$$

$\Delta < 0$, so this equation has **no real roots**.

You may also be told what the nature of the roots are when the equation contains an unknown coefficient – such as $2x^2 + px + 1$ or $ax^2 - x - 3$, and you would then be expected to solve an equation to work out what p or a could be.

Example 2 – range of values

Find the range of values of p such that the equation $3x^2 - 6x + p = 0$ has real and distinct roots.

Solution

For real and distinct roots, the discriminant must be positive, so we solve $\Delta > 0$.

For this equation, $a = 3$, $b = (-6)$ and $c = p$. We can now work out Δ .

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (-6)^2 - 4\times 3\times p\\ &= 36 - 12p\end{aligned}$$

Since we know the discriminant is $36 - 12p$ and that the discriminant must be positive, we can solve the equation:

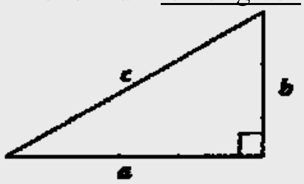
$$\begin{aligned}36 - 12p &> 0\\ -12p &> -36\\ p &< \tfrac{-36}{-12} \qquad \text{(dividing by a negative means > changes to <)}\\ p &< 3\end{aligned}$$

Lengths and Angles

Pythagoras' Theorem

At National 4, you learnt how to use Pythagoras' Theorem to find the length of the third side in a right-angled triangle without needing to measure it.

Theorem of Pythagoras. This formula is **not** given on the National 5 exam paper.



$$a^2 + b^2 = c^2$$

There are three steps to any Pythagoras question:

Step One – square the length of the two sides

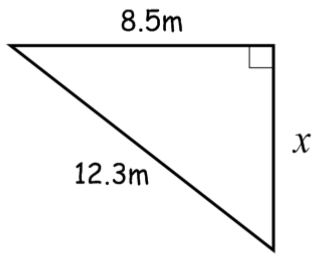
Step Two – either add or take away

- To find the length of the longest side (the hypotenuse), **add** the squared numbers.
- To find the length of a shorter side, **take away** the squared numbers.

Step Three – square root

Example 1 – reminder of the basic method from National 4

Calculate x, correct to 1 decimal place.



Solution

We are finding the length of x. x is s smaller side, so we **take away**.

$$x^2 = 12.3^2 - 8.5^2$$
$$x^2 = 79.04$$
$$x = \sqrt{79.04}$$
$$x = 8.8904....$$
$$x = \underline{8.9\,\text{m}} \text{ (1d.p.)}$$

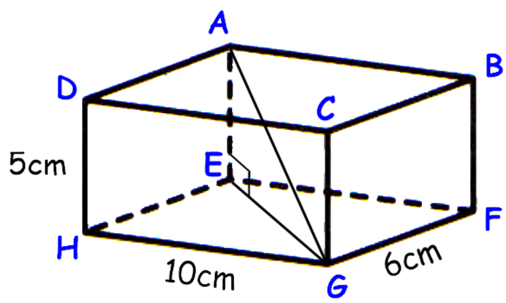
At National 5 you need to be able to extend your ability to use Pythagoras in more challenging situations:

- in **three-dimensional** diagrams
- in diagrams involving **circles**
- using the **converse** of Pythagoras

Example 2 – Pythagoras in 3d

The diagram shows a cuboid with sides of length 5cm, 10cm and 6cm.

Calculate the length of the diagonal AG.



Solution

To find AG, we have to use Pythagoras in triangle AEG. However before we can do this, we must calculate length EG.

First we use Pythagoras in triangle EGH, in which EG is the hypotenuse, and the other sides are 10cm and 6cm.

$$\begin{aligned} EG^2 &= 10^2 + 6^2 \\ &= 136 \\ EG &= \sqrt{136} \end{aligned}$$

We could write this answer as a decimal but it is actually easier for the next step to leave it as a surd.

Now we use Pythagoras in triangle AEG, which has hypotenuse AG, and other sides of length 5cm, and $\sqrt{136}$ cm.

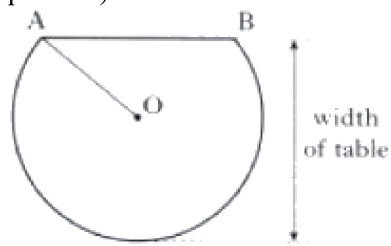
$$\begin{aligned} AG^2 &= \left(\sqrt{136}\right)^2 + 5^2 \\ &= 161 \\ AG &= \sqrt{161} \\ &= \underline{\underline{12.7\text{cm}}} \text{ (1d.p.)} \end{aligned}$$

Definition: a **chord** in a circle is a straight line going from one side of the circle to another.

Example 3 – Pythagoras in circles (past Intermediate 2 exam question)

A table top is in the shape of part of a circle.
The centre of the circle is O.
AB is a chord of the circle.
AB is 70cm. The radius OA is 40cm.

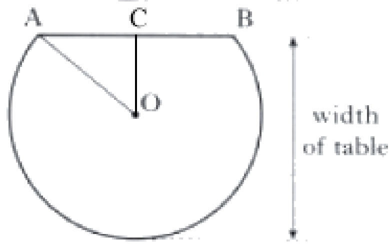
Find the width of the table.



Solution

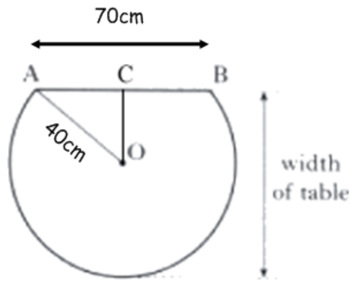
Step one – draw a new line in the diagram to make the diagram symmetrical.

In the diagram on the right, this is OC. This means that triangle OAC is right-angled.



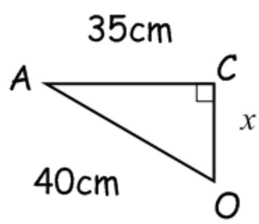
Step two – write in the length of the radius and any other lengths we are told.

OA is 40cm and AB is 70cm.



Step three – identify the lengths in the right-angled triangle, using the fact that the 'new' line splits the chord in half.

In this diagram, AB is 70cm, so AC must be 35cm.



Step four – use Trigonometry or Pythagoras to calculate another length (or angle) in the triangle.

Using Pythagoras:

$$\begin{aligned} x^2 &= 40^2 - 35^2 \\ &= 375 \\ x &= \sqrt{375} \\ &= \underline{19.4\text{cm (1d.p.)}} \end{aligned}$$

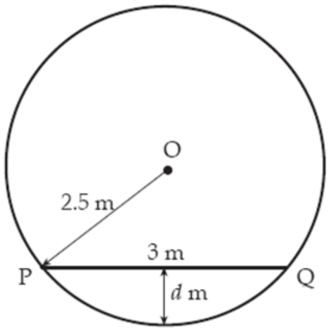
Step five – check whether you have answered the whole question

In this question, they wanted the width of the table. The width of the table is OC + the radius. So in this diagram, the width is 19.4 + 40 = 59.4cm.

Example 4 – Pythagoras in circles (past Intermediate 2 exam question)

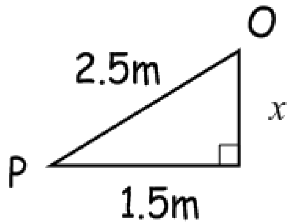
The diagram shows the cross-section of a cylindrical oil tank. The centre of the circle is O.
PQ is a chord of the circle. PQ is 3m.
The radius OP is 2.5m.

Find the depth, *d*, of the oil.



Solution

Steps one, two and three – (see previous example for more detail on how to do these steps) identify the lengths in the right-angled triangle, using the fact that the diagram is symmetrical.



Step four – use Trigonometry or Pythagoras to calculate another length (or angle) in the triangle.

Using Pythagoras:

$$\begin{aligned} x^2 &= 2.5^2 - 1.5^2 \\ &= 4 \\ x &= \sqrt{4} \\ &= \underline{2\text{m}} \end{aligned}$$

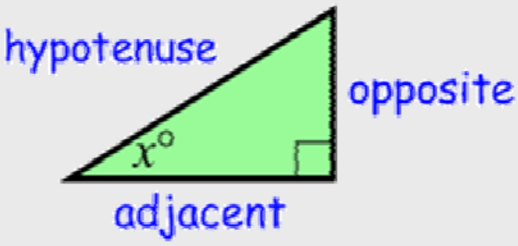
Step five – check whether you have answered the whole question

In this question, they wanted the depth of the water. From the diagram, *d* and *x* add together to make a radius (2.5m). Since *x* = 2 metres, *d* must be 0.5 metres.

Right-Angled Trigonometry (SOH CAH TOA): A Revision

At National 4, you learnt to use sine, cosine and tangent in a right-angled triangle. At National 5 you may be expected to remember these skills as part of a longer question. The formulae and basic method are included here as a reminder.

Trigonometric ratios in a right angled triangle:
Formula. These formula <u>are **not** given</u> on the National 5 exam paper.



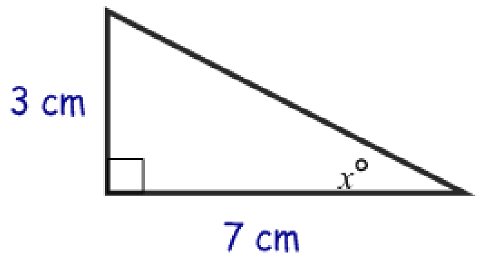
$$\sin x° = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos x° = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan x° = \frac{\text{Opposite}}{\text{Adjacent}}$$

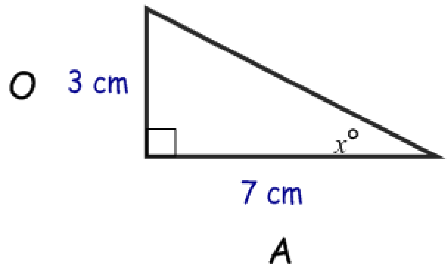
<u>Example 1 – calculating an angle</u>

Find the size of angle $x°$ in this right-angled triangle.



Solution

Step one – label the sides O, H or A. Do not bother labelling the side that we do not know the length of (in this case it is H), as we don't need it.



Step two – here we have the **opposite** and the **adjacent**. Now use SOH CAH TOA to tick off the sides we have:

√ √ √√
SOH CAH TOA

TOA has two ticks above it, telling us to use **tan** in this question.

Step three – copy out the formula carefully

Step four – substitute the numbers 3 and 7 in the correct places

Step five – use $\tan^{-1}$ (or $\sin^{-1}$ or $\cos^{-1}$) to find the angle

$$\tan x = \frac{\text{opposite}}{\text{adjacent}}$$

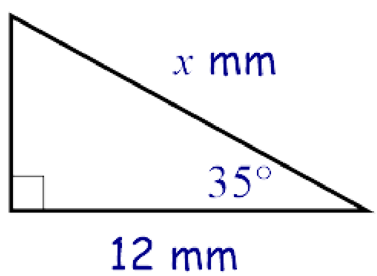
$$\tan x = \tfrac{3}{7}$$

$$x = \tan^{-1}(3 \div 7)$$

$$x = 23.198\ldots = \underline{23.2°}\ (\text{1d.p.})$$

<u>Example 2 – calculating a length</u>

Find the length x in this right-angled triangle.



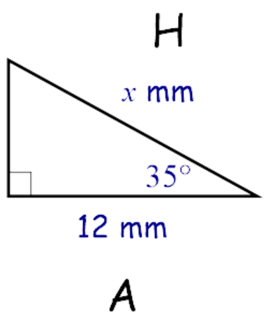
Solution

Step one – label the sides O, H or A. One side will have nothing written on it – do not bother labelling that side (in this case it is O) as we do not need it.

Step two – we have the **adjacent** and the **hypotenuse**. Now use SOH CAH TOA to tick off the sides you have:

√ √√ √

SOH CAH TOA



CAH has two ticks above it, telling us to use **cos** in this question.

Step three – copy out the formula carefully

$$\cos x = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Step four – substitute the numbers 35 and 12 in the correct places. ***The angle always has to go straight after sin (or cos or tan)***

$$\cos 35 = \frac{12}{x}$$

Step five – to move the x over to the left hand side you multiply

$$x\cos 35 = 12$$

Step six – to move the cos 35 over to the other side you divide

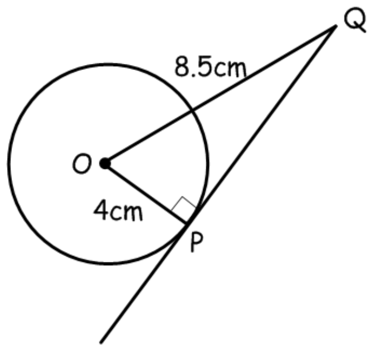
$$x = \frac{12}{\cos 35}$$

$$x = 14.649... = \underline{14.65\text{mm}} \text{ (2d.p.)}$$

If there is a right-angle inside a triangle in a diagram, then you can use Pythagoras or trigonometry to work out other lengths or angles inside that triangle.

<u>Example 3 – diagram involving a circle</u>

PQ is a tangent to a circle, centre O. Calculate angle POQ.



Solution

The fact that PQ is a tangent means that the angle OPQ is 90°. This means that OPQ is a right-angled triangle with the hypotenuse 8.5cm and the adjacent side 4cm. Using the standard method:

$$\cos POQ = \frac{4}{8.5}$$
$$POQ = \cos^{-1}(4 \div 8.5)$$
$$POQ = \underline{61.9°}$$

Angles

You will be expected to work out angles in diagrams using the properties of angles. :

- Two angles on a straight line add up to make 180°.
- Opposite angles in **X-shapes** are equal.
- The three angles in a **triangle** add up to make 180°.
  - And in particular, **Isosceles triangles** are symmetrical, meaning that the two angles in the base are the same size.
- The four angles in a **quadrilateral** add up to make 360°.
- Angles in **Z shapes** (made by parallel lines) are the same.
- A **tangent line** to a circle meets a radius at right-angles.
- The **angle in a semicircle** is a right angle.

Questions about angles in the exam require you to identify the right-angles and isosceles triangles, and then using the rules of angles to find any remaining angles.

You can only write a right angle in a diagram if you *know* it is a right angle. There are some occasions when you can *know* an angle is a right-angle:

1. If you are told that a shape is a **square** or **rectangle**, you know the angles in its corners are right angles.
2. A **tangent** always makes a right angle with the radius.
3. Triangles in **semi-circles** are always right angled.
4. If a line in a circle diagram is a line of symmetry, it will cross any other line at right-angles.

Essential Exam Tips

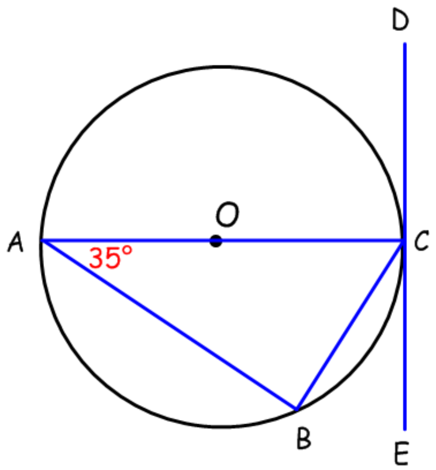
- Always copy (or trace) the diagram onto your answer sheet and mark in all angles clearly.
- Make sure you make it clear which angle is your final answer – writing it on the diagram isn't enough unless you indicate clearly which angle is the one we are looking for.

Two examples are given below – however every question is different. The only way to get used to them is to practice them from past exam papers and textbooks.

Example 1 – angles and circles

The diagram shows a circle centre O. DE is a tangent to the circle at point C. Angle OAB is 35°.

Calculate the size of angle BCE



Solution

Copy the diagram onto your exam paper, and mark each angle in turn.

A tangent and a radius meet at right angles, so ACD and ACE are 90°.

The angle in a semi-circle is a right angle, so ABC is also 90°.

We know two of the angles in triangle ABC. Since we know the angles in a triangle must add to make 180°, angle ACB must be 55°.

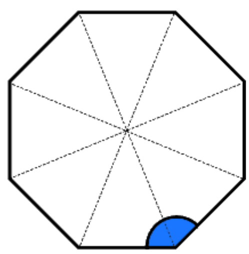
Finally since we already knew ACE is 90°, this tells us that ACB and BCE must add to make 90°. Therefore angle BCE is 35°.

Final answer: clearly state that angle BCE is 35° (just marking it on the diagram isn't enough as it doesn't make it clear that you know which angle is angle BCE).

You may be expected to work out angles inside other shapes, usually regular polygons (i.e. straight sided shapes where all sides and angles are the same). The key to these questions is to split the shape up into identical isosceles triangles.

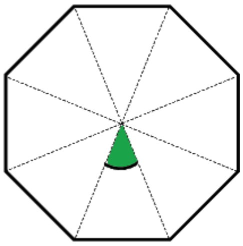
Example 2 – regular polygons

The diagram shows a regular octagon. Calculate the size of the shaded angle.



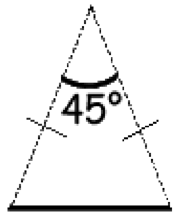
Solution

In the diagram, the octagon has been divided into eight smaller triangles. Since the octagon is regular, each triangle is the same size.



Step one – work out the angle in the centre (as pictured)

Eight equal triangles fit together to make 360°, so the shaded angle is 360 ÷ 8 = 45°



Step two – we now have an isosceles triangle with one angle 45°. We now work out the other two (equal) angles.

180 – 45° = 135°
135 ÷ 2 = 67.5° .

Step three – to find the shaded angle in the original diagram, we double the angle we just found. 67.5° × 2 = 135°

Similar Shapes

Two shapes are similar if they are exactly the same shape but a different size. All their angles will be identical, and all their lengths will be in proportion to each other. One shape will be an enlargement of the other. The factor of the enlargement is called the **scale factor**.

$$\text{Scale Factor} = \frac{\text{Length in 'new' shape}}{\text{Length in 'old' shape}}$$

For National 5, you need to understand how the scale factor affects area and volume.

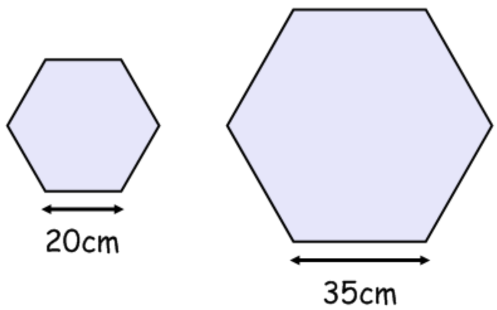
Facts

For two similar shapes connected by a scale factor *s*:

- Lengths in the 'new' shape are found by multiplying lengths in the 'old' shape by *s*
- The area of the 'new' shape is found by multiplying the area of the 'old' shape by s^2
- The 'new' volume is found by multiplying the volume of the 'old' shape by s^3

Example 1 – area

Two hexagons are similar in shape as shown in the diagram. The smaller hexagon has an area of 2500cm. Calculate the area of the larger hexagon.



Solution

Step one – calculate the scale factor.

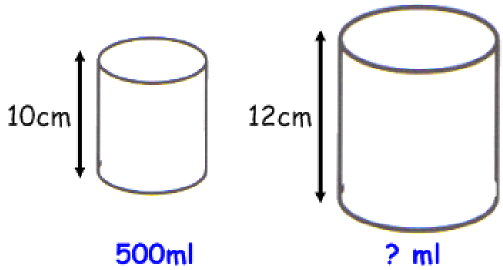
$$\text{s.f.} = \frac{35}{20} = 1.75$$

Step two – calculate the area, remembering to *square* the scale factor for area.

$$\text{Area} = 2500 \times 1.75^2 = \underline{\underline{7656.25\,\text{cm}^2}}$$

Example 2 – volume

Two cylindrical drinks cans are mathematically similar. The smaller can holds 500ml of juice. How much will the larger can hold?



Solution

Step one – calculate the scale factor.

$$\text{s.f.} = \frac{12}{10} = 1.2$$

Step two – calculate the volume, remembering to *cube* the scale factor for volume.

$$\text{Area} = 500 \times 1.2^3 = \underline{\underline{864\text{ml}}}$$

Example 3 – backwards

Two rectangles are mathematically similar. The area of the larger rectangle is double the area of the smaller rectangle. If the breadth of the smaller rectangle is 10cm, find the breadth of the larger rectangle.

Solution

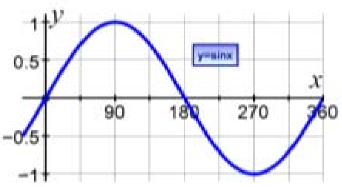
The scale factor connecting the areas is s^2. The information in the question tells us that $s^2 = 2$. Solving this equation tells us that s must be $\sqrt{2}$.

Therefore the breadth of the larger rectangle is $10 \times \sqrt{2} = \underline{\underline{14.1\text{cm}}}$ (1d.p.)

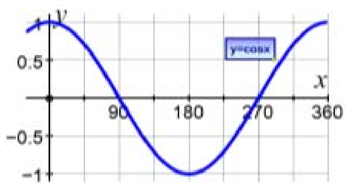
Trigonometry

Graphs of sin, cos and tan

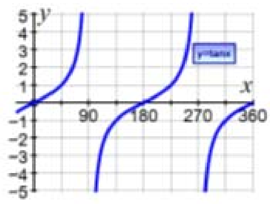
You are should know what the graphs of sin x, cos x and tan x look like between 0° and 360°:



sin



cos



tan

Definition: the **frequency** of a sin or cos graph is how many times the graph repeats itself in 360°. The frequency of a tan graph is how many times it repeats itself in 180°. <u>In the equation of a sin, cos or tan graph, the frequency is the number before *x*.</u>

Definition: the **amplitude** is a way of describing the height of a sin or cos graph – e.g. the sine and cosine graphs above both have an amplitude of 1. <u>In the equation of a sin or cos graph, the amplitude is the number before sin or cos.</u>

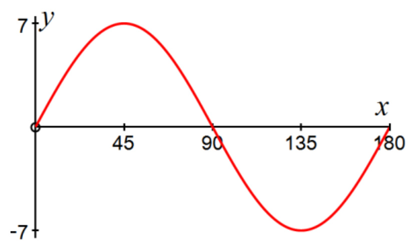
Definition: the **period** of a graph is how many degrees it takes the graph to do one complete cycle. In the graphs above, sin x and cos x have a period of 360° and tan x has a period of 180°.

$$\text{Period of a sin or cos graph} = \frac{360°}{\text{Frequency}} \qquad \text{Period of a tan graph} = \frac{180°}{\text{Frequency}}$$

Equation	Frequency	Amplitude	Period
$y = \cos x$	1	1	360°
$y = 3\sin 4x$	4	3	90°
$y = 6\cos 2x$	2	6	180°
$y = 5\tan 2x$	2	5	90°

<u>Example 1</u>

The graph on the right has an equation of the form $y = a\sin bx$. What are the values of a and b?



Solution

The maximum and minimum are 7 and –7, so the amplitude is 7 – i.e. a = 7.

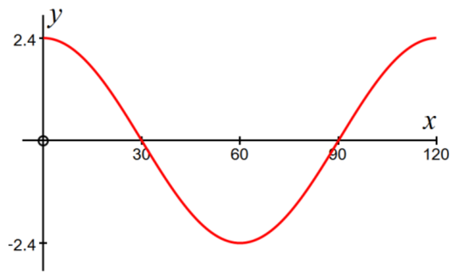
The graph repeats itself once between 0° and 180°. This means that it repeats itself twice between 0° and 360°, so it has a frequency of 2, so $b = 2$.

Answer: the graph is $\underline{y = 7\sin 2x}$

Look at the x axis carefully. It is not just a case of counting how many waves you can see, but you have to work out how many there would be in 360°. In exams, the graphs will often stop before 360°.

Example 2

The graph on the right has an equation of the form $y = a\cos bx$. What are the values of a and b?



Solution

The maximum and minimum are 2.4 and –2.4, so the amplitude is 2.4. This means that a = 2.4.

This graph only goes up to 120°. The graph repeats itself once between 0° and 120°. This means that it repeats itself three times between 0° and 360°, so its frequency is 3, meaning b = 3.

Answer: the graph is $y = 2.4\cos 3x$

Example 3

What is the period of the function $3\cos 4x$?

Solution

From the equation, the frequency of the graph is 4.

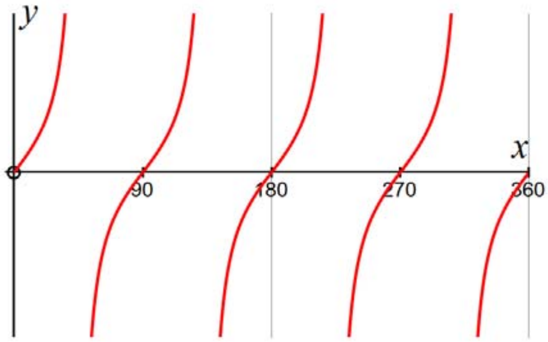
Using the formula for period, the period is 360 ÷ 4 = 90°

With a tan graph, the frequency is how many times the graph repeats in 180°, not 360°.

Example 4

The graph on the right has an equation of the form $y = \tan bx$.

What is the values of b?



Solution

The frequency of a tan graph is the number of waves in 180°.

This means the frequency of this graph is 2 (and not 4), so b = 2.

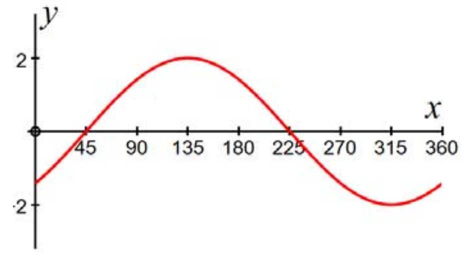
Answer: the graph is $y = \tan 2x$

Definition: the **phase angle** is the amount a graph has been shifted to the right. In the equation of a sin, cos or tan graph, the phase angle is the number taken away from x in the brackets.

Equation	Period	Amplitude	Phase Angle
$y = 2\cos(x-45)°$	1	2	45°
$y = \sin(x-30)°$	1	1	30°
$y = 4\cos(2x-15)°$	2	4	15°

Example 5

The graph on the right has an equation of the form $y = a\sin(x - b)°$. What are the values of a and b?



Solution

The maximum and minimum are 2 and –2, so the amplitude is 2. So this means that $a = 2$.

The graph has been shifted 45° to the right, so $b = 45°$.

Answer: the graph is $y = 2\sin(x - 45)°$

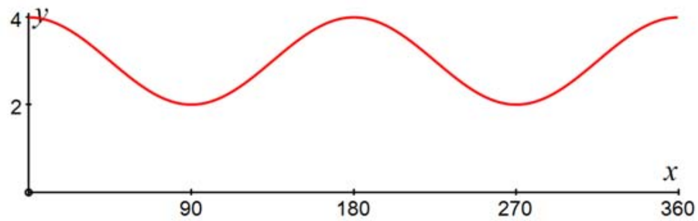
Definition: a **vertical translation** is the amount a graph has been shifted up or down. In the equation of a sin, cos or tan graph, the vertical translation is the number added on at the end of an equation.

Equation	Amplitude	Vertical translation
$y = 2\cos x + 3$	2	3
$y = 5\sin 2x - 1$	5	–1

Example 6

The graph on the right has an equation of the form $y = \cos bx + c$.

What are the values of b and c?



Solution

b is the frequency. The graph repeats twice in 360°, so this means that $b = 2$.

A cos graph would normally have y values between 1 and –1. This graph goes between 4 and 2. This means all the values are 3 higher. Therefore $c = 3$.

Answer: the graph is $y = \cos 2x + 3$

Sin, cos and tan without a calculator

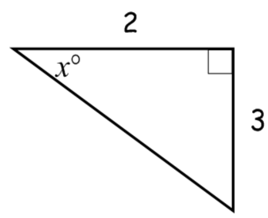
Recall the original definitions of sin, cos and tan from National 4:

These formulae are **not** given on the National 5 exam paper.

$$\sin x = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos x = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan x = \frac{\text{Opposite}}{\text{Adjacent}}$$

Example 1

Write down the value of sin x in this triangle



Solution

We need to use the formula $\sin x = \frac{\text{Opposite}}{\text{Hypotenuse}}$.

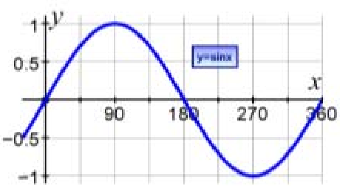
From the diagram, the opposite side is 3.

We do not know the hypotenuse, but we can work it out using Pythagoras:

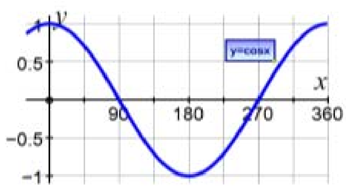
$$H^2 = 2^2 + 3^2$$
$$= 13$$
$$H = \sqrt{13}$$

Opposite = 3 and Hypotenuse = $\sqrt{13}$, so $\sin x = \dfrac{3}{\sqrt{13}}$

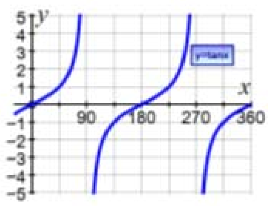
In some questions, you need to use the knowledge of whether sin, cos or tan is negative for a particular angle. To know whether they are positive or negative, you can look at the graphs:



sin



cos



tan

Another way of working out is to use a CAST diagram, which allows us to identify **related angles**. Related angles are angles that have the same value of sine, cos or tan (or at least they are the same except for positive or negative signs).

The angles they ask you about in an exam question will always be related; and your main task is to work out whether the sin, cos or tan value is positive, negative or zero.

Example 2

Given that $\cos 60° = 0.5$, write down the value of $\cos 120°$

Solution

From the graph of cos (or a CAST diagram), cos 120° is negative.

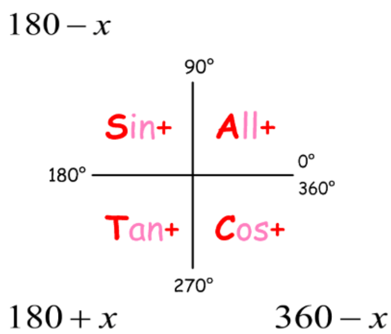
Answer: cos 120° = –0.5

Solving Trigonometric Equations (Equations with sin, cos and tan in)

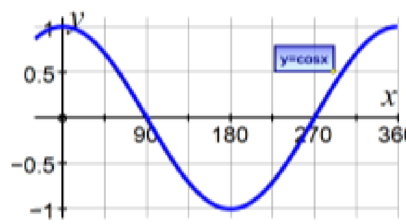
You can recognise these questions because they will ask you to **solve each equation for** $0 \le x < 360$**.** (This just means *x* must between 0° and 360°.)

These questions will usually have TWO answers:

- Your calculator will give you the first using $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$
- To get the other, you need either a CAST diagram or a sketch of the graph.



or



Example 1 (Positive values of sin, cos and tan)

Solve the equation $5\sin x - 2 = 1$ for $0 \le x < 360°$:

Solution:

Step One – rearrange the equation

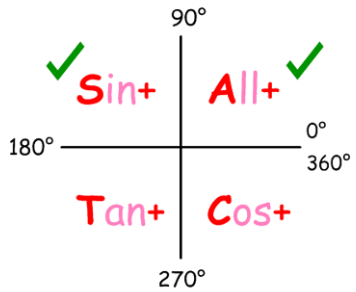
$$5\sin x - 2 = 1$$
$$5\sin x = 1 + 2$$
$$5\sin x = 3$$
$$\sin x = \tfrac{3}{5}$$

Step Two – find the first solution using $\sin^{-1}$

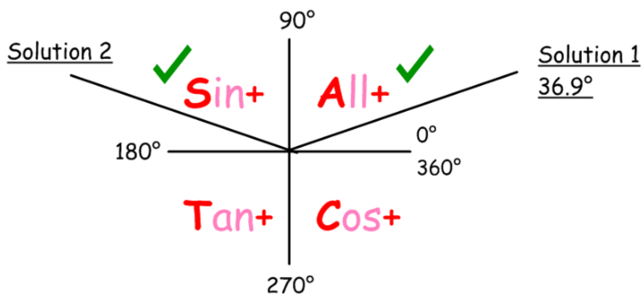
$$x = \sin^{-1}(3 \div 5)$$
$$x = 36.9°$$

Step Three – find the second solution using CAST

This question involves **sin**. The number on the right is $\frac{3}{5}$, which is **positive**. This means we tick the ALL and SIN quadrants.



Putting both answers into the diagram shows using symmetry that solution 2 is given by 180 – 36.9 = 143.1°.



Answer: $x = 36.9°$, $x = 143.1°$

Example 2 (Negative values of sin, cos and tan)

Solve the equation $3\cos x + 3 = 1$ for $0 \le x < 360°$:

Solution:

Step One – rearrange the equation

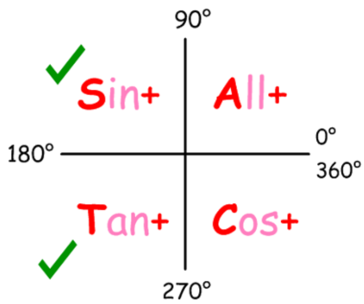
$$3\cos x + 3 = 1$$
$$3\cos x = 1 - 3$$
$$3\cos x = -2$$
$$\cos x = \tfrac{-2}{3}$$

Step Two – find the first solution using $\cos^{-1}$

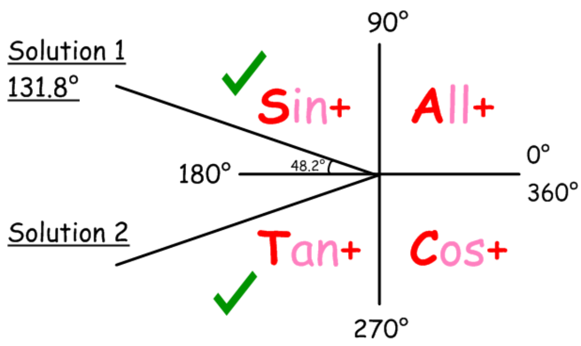
$$x = \cos^{-1}(-2 \div 3)$$
$$x = 131.8°$$

Step Three – find the second solution using CAST

This question involves **cos**. The number on the right is $\frac{-2}{3}$, which is **negative**. This means that cos is NOT positive, so we do NOT tick ALL and COS, instead we tick the SIN and TAN quadrants.



Putting both answers into the diagram shows using symmetry that solution 2 is given by 180 + 48.2 = 228.2°.

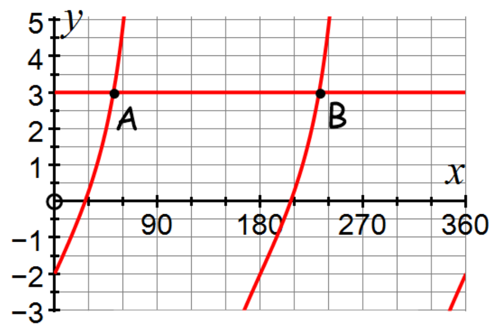


Answer: $x = 131.8°$, $x = 228.2°$

You may be asked to find the point of intersection of two graphs. To do this, we form an equation by making the equations of each graph equal to each other. For example to find the points of intersection of the graphs $5\sin x$ and $y = 3$, we would solve the equation $5\sin x = 3$.

Example 3 – from a graph

The diagram shows the graphs of $y = 4\tan x - 2$ and the graph of $y = 3$ for $0 \le x < 360°$. Find the *x*-coordinates of the points of intersection A and B.



Solution

To find the points of intersection, we make the two equations equal to each other: i.e. we solve the equation $4\tan x - 2 = 3$.

We now solve this using the method from examples 1 and 2:

Step One – rearrange the equation

$$4\tan x - 2 = 3$$
$$4\tan x = 3 + 2$$
$$4\tan x = 5$$
$$\tan x = \tfrac{5}{4}$$

Step Two – find the first solution using $\tan^{-1}$

$$x = \tan^{-1}(5 \div 4)$$
$$x = 51.3°$$

Step Three – find the second solution using CAST

This question involves **tan**. The number on the right is $\tfrac{5}{4}$, which is **positive**. This means we tick the ALL and TAN quadrants.

A CAST diagram shows us that solution 2 is given by 180 + 51.3 = 231.3°.

Answer: The *x*-coordinates are $x = 51.3°$ and $x = 231.3°$

Trigonometric Identities

Formula. These formulae are **not** given on the National 5 exam paper.

$$\tan x = \frac{\sin x}{\cos x} \qquad \sin^2 x + \cos^2 x = 1$$

We use trigonometric identities to simplify more complex expressions. A question would normally ask you to "Prove that..." a fact about sin, cos or tan is true or to "simplify" an expression involving sin, cos or tan.

There are not really any set rules for how to do these questions – instead you have to use your mathematical ability to choose the correct rules of algebra to find the answer. However the following tips are a good starting point:
Tip 1 – only ever "do stuff" to the more complicated expression (often the one on the left-hand side). Leave the simpler expression alone.

Tip 2 – you only get marks for knowing and using the two formulae in the grey box above. So a good bit of advice is to look at the more complicated expression, and:

a) If you see ' $\tan x$ ' on the left hand side, replace it with $\dfrac{\sin x}{\cos x}$
b) If you see ' $\dfrac{\sin x}{\cos x}$ ' on the left hand side, replace it with $\tan x$
c) If you see ' 1 ' on the left hand side, replace it with $\sin^2 x + \cos^2 x$
d) If you see ' $\sin^2 x + \cos^2 x$ ' on the left hand side, replace it with 1

Example 1

Prove that $\mathbf{\dfrac{1-\cos^2 x}{3\sin^2 x} = \dfrac{1}{3}}$

Solution

Using Tip 1: The left-hand side is more complicated, and the right-hand side is simpler. This means that we will "do stuff" to the left-hand side $\dfrac{1-\cos^2 x}{3\sin^2 x}$.

Using Tip 2: Using tip 2c above, we replace '1' with $\sin^2 x + \cos^2 x$:

$$\frac{1-\cos^2 x}{3\sin^2 x} = \frac{\sin^2 x + \cos^2 x - \cos^2 x}{3\sin^2 x}$$

We can now do some simplifying

$$\frac{\sin^2 x \cancel{+\cos^2 x} \cancel{-\cos^2 x}}{3\sin^2 x} = \frac{\sin^2 x}{3\sin^2 x} = \frac{{}^1\cancel{\sin^2 x}}{3\cancel{\sin^2 x}} = \frac{1}{3}$$, which is what we wanted

---

Example 2

Simplify $\mathbf{5\sin^2 x + 5\cos^2 x}$

Solution

Tip 1 isn't relevant here as there is only one expression.
Using **Tip 2**, none of those four expressions appear on the left-hand side exactly. However if we spot there is a common factor of 5 and <u>factorise it first</u>, we can use tip 2d:

$$\begin{aligned} 5\sin^2 x + 5\cos^2 x &= 5(\sin^2 x + \cos^2 x) \\ &= 5(1) \\ &= 5 \end{aligned}$$

. **Answer:** <u>5</u>

Applications Unit

Trigonometry

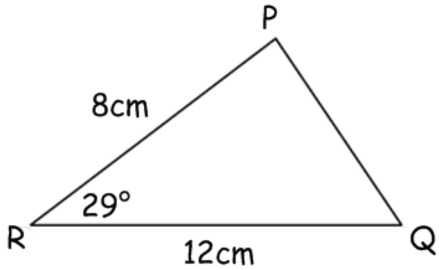
Area of a triangle

To find the area of any triangle you need the length of two sides and the size of the angle between them.

Formula. This formula is given on the National 5 exam paper.
Area of a Triangle: $A = \frac{1}{2}ab\sin C$

Important: in the formula, *a* and *b* mean the lengths, and *C* means the angle. It is possible that you may be given a diagram where different letters are used. You have to ignore these letters and relabel the two sides *a* and *b* (order does not matter) and the angle in between them as *C*.

Example
Find the area of this triangle. Round your answer to 3 significant figures.



Solution
a and *b* are the lengths, so we use $a = 8$, $b = 12$
C is the angle between the lengths, so $C = 29°$

$A = \frac{1}{2}ab\sin C$

$A = 8 \times 12 \times \sin 29° \div 2$

$A = 23.27086177...$

$A = 23.3\text{cm}^2$ (3 s.f.)

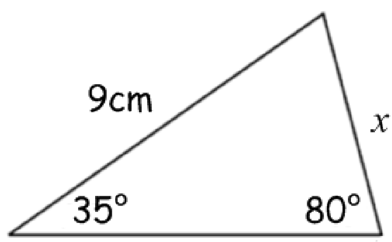
Sine Rule

Formula. This formula is given on the National 5 exam paper.
Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Where *a*, *b* and *c* are the lengths of the sides of the triangle, and *A*, *B* and *C* are the angles in the triangle. Side *a* is opposite angle *A* etc.

Important: to answer a question you do not use the formula as it is written. You only need the first two 'bits': $\frac{a}{\sin A} = \frac{b}{\sin B}$

Example 1 – sine rule for lengths
Find the length *x* in this triangle.



Solution

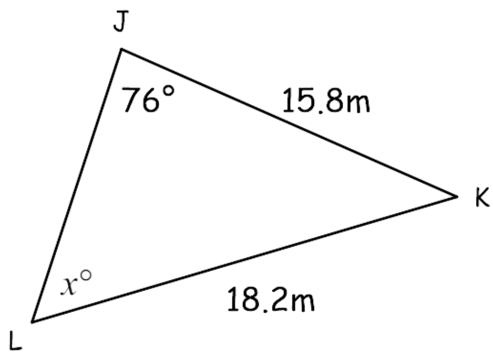
x cm is opposite 35°, so use $a = x$ and $A = 35°$
9cm is opposite 80°, so use $b = 9$ and $B = 80°$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{x}{\sin 35} = \frac{9}{\sin 80}$$
$$x = \frac{9\sin 35}{\sin 80} \qquad \text{(moving the } \sin 35 \text{ to the other side)}$$
$$x = 5.241822996...$$
$$\underline{x = 5.2\text{cm (1 d.p.)}}$$

If there are *two triangles* in the diagram, you have to use the sine rule (or cosine rule, or SOH CAH TOA) twice. These questions will normally have 4-6 marks.

Example 2 – sine rule for angles

Calculate the size of angle x° in this triangle



Solution

x° is opposite 15.8°, so use a = 15.8 and A = x°
76° is opposite 18.2m, so use b = 18.2 and B = 76°

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{15.8}{\sin x} = \frac{18.2}{\sin 76}$$
$$15.8\sin 76 = 18.2\sin x \qquad \text{(cross-multiplying)}$$
$$\sin x = \frac{15.8\sin 76}{18.2} \qquad \text{(dividing by 18.2 to make } \sin x \text{ the subject)}$$
$$\sin x = 0.842344...$$
$$x = \sin^{-1} 0.842344...$$
$$x = 57.3885...$$
$$\underline{x = \underline{\underline{57.4}}° \text{ (1 d.p.)}}$$

The Cosine Rule

Formula. This formula is <u>given</u> on the National 5 exam paper.

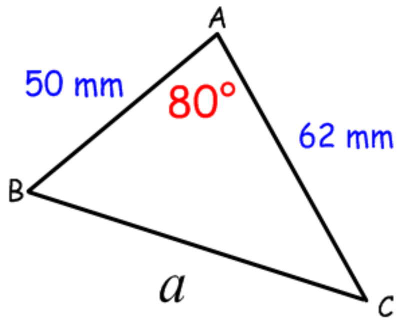
Cosine rule: $a^2 = b^2 + c^2 - 2bc\cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

where A is the angle <u>between</u> the two sides b and c, and a is the length opposite angle A. You use the first version of the formula to calculate a *length*, and the second to calculate an *angle*.

To find a **length** using the cosine rule, you <u>must know</u> the other two sides and the angle in between. It does not matter which side is called b and which is called c.

Example 1

Find the length of *a* in this diagram



Solution

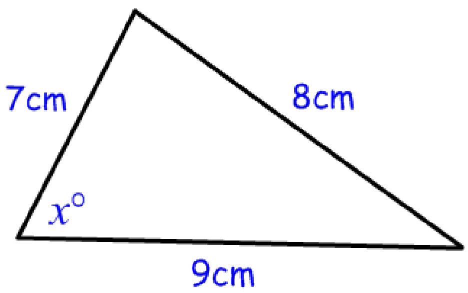
$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 50^2 + 62^2 - 2\times 50\times 62\times\cos 80 \\ &= 6344 - 6200\cos 80 \\ &= 5267.3881298... \\ a &= \sqrt{5267.388...} \\ &= 72.5767... \\ &= \underline{72.6\text{mm}}\ (1\text{ d.p.}) \end{aligned}$$

To find an **angle** using the cosine rule, you must know the lengths of all three sides to be able to use this formula. To find an angle, you use the second version of the formula.

In the formula $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, it is crucial that *a* must be the side opposite the angle you are finding. It does not matter which way around *b* and *c* go.

Example 2 – finding an angle

Find the size of angle *x*° in this diagram



Solution

Length '*a*' has to be the side opposite the angle we are finding, so $a = 8$.

It does not matter which way around *b* and *c* go, so we will say $b = 7$ and $c = 9$.

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ \cos x &= \frac{7^2 + 9^2 - 8^2}{2\times 7\times 9} \\ &= \frac{66}{126} \\ x &= \cos^{-1}\left(\tfrac{66}{126}\right) \\ &= 58.411864... \\ &= \underline{58.4°}\ (1\text{ d.p.}) \end{aligned}$$

Choosing the Formula

Use the key words in the question to decide if you are being asked to calculate the area, an angle, or a length. Use the diagram to see what information you are being given.

Keywords: what is the question asking you to find?

- **AREA:** use $A = \frac{1}{2}ab\sin C$

- **ANGLE:**
  - if you know all three sides use the **cosine rule for angles** $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$
  - if you only know two sides, use the **sine rule** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

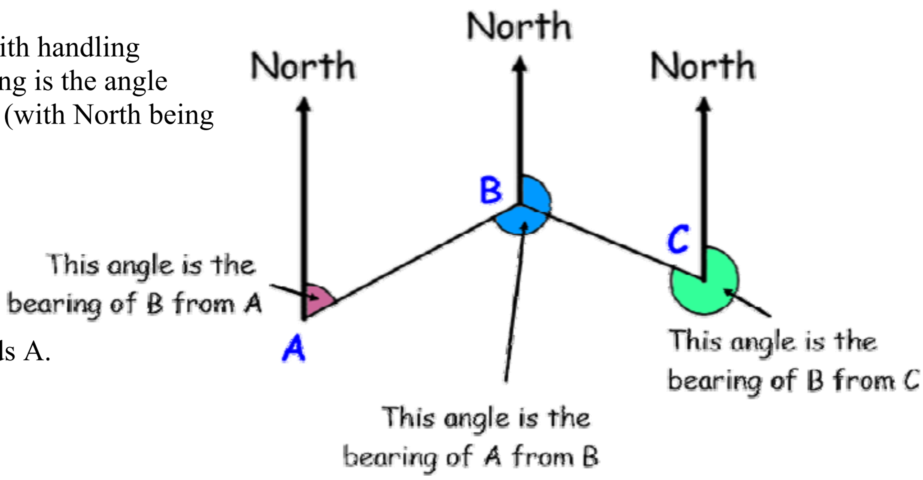
- **LENGTH (distance, how far, how long, etc):**
  - if you know two sides and the angle in between, use the **cosine rule** ($a^2 = b^2 + c^2 - 2bc\cos A$)
  - if you know at least two angles and at least one other length, use the **sine rule** $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

As a rough rule, if you know (or can work out) all of the angles in the triangle, you are probably going to use the sine rule.

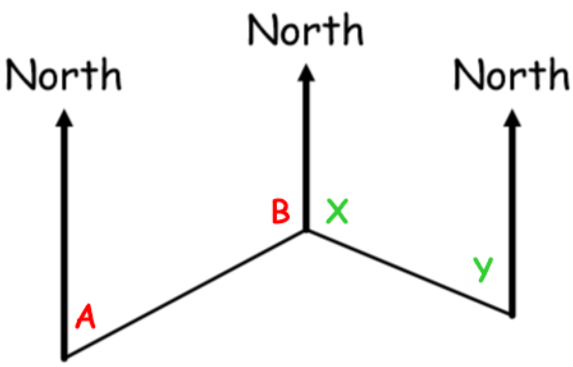
Bearings

You also need to be confident with handling bearings in diagrams. The bearing is the angle measured clockwise from North (with North being 000°)

When describing bearings, it is important to go the correct way. The bearing of A **from** B refers to the angle that is centred at B and pointing towards A.



Fact
In a diagram involving bearings, some angles add to make 180°. In this diagram, A and B add to make 180°, and so do X and Y. The other bearing is referred to as the **back bearing** – e.g. A is the back bearing of B.

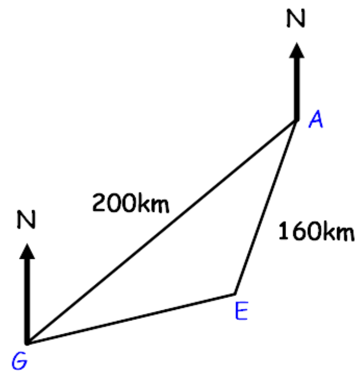


Example (past SQA Standard Grade Credit question)

The diagram shows the position of airports A, E and G.

- **G is 200 kilometres from A**
- **E is 160 kilometres from A**
- **From G the bearing of A is 052°**
- **From A the bearing of E is 216°**

How far apart are airports G and E?

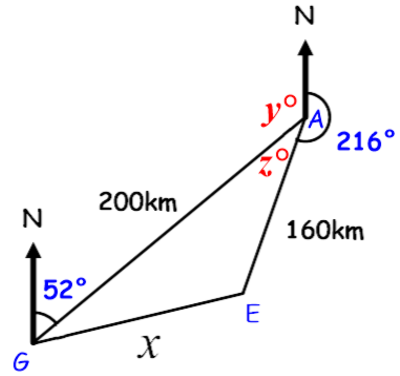


Solution

Step one – put the information from the question into the diagram

We begin by adding the angles to the diagram.

Also since the distance we are being asked to find is GE, we can label that side as x.

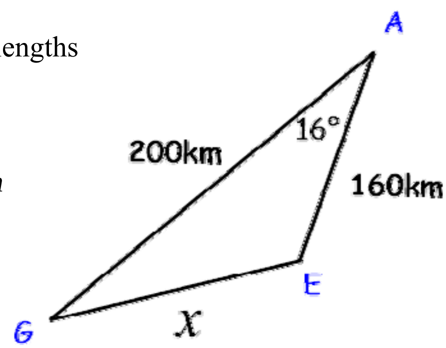


Step two – use the rules of angles to fill in the other angles in the diagram

The angle marked y is the back bearing, so
$$\begin{aligned} 52 + y &= 180 \\ y &= 180 - 52 \\ &= 128° \end{aligned}$$

The angle marked 216°, along with the angles called y and z add up to make 360°. Therefore
$$\begin{aligned} 216 + y + z &= 360 \\ 216 + 128 + z &= 360 \\ z &= 360 - 216 - 128 \\ &= 16° \end{aligned}$$

Step three – sketch triangle AEG, showing the angles and lengths



Step four – use the sine rule or cosine rule to solve for x

In this triangle, we have two sides and the angle in between, so we choose the cosine rule:

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ &= 200^2 + 160^2 - 2 \times 200 \times 160 \times \cos 16 \\ &= 4079.251... \\ a &= \sqrt{4079.251...} \\ &= \underline{63.9\text{km}} \text{ (1 d.p.)} \end{aligned}$$

Vectors

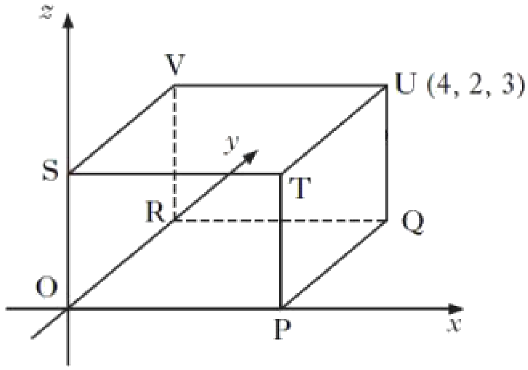
3d Coordinates

We can extend the traditional 2 dimensional Cartesian diagram into 3 dimensions by adding a third axis called the *z* axis which is at right angles to both the *x* axis and *y* axis.

Example (diagram adapted from 2010 Higher exam paper)

In the diagram on the right, the point U has coordinates (4, 2, 3).

State the coordinates of P, V and Q.

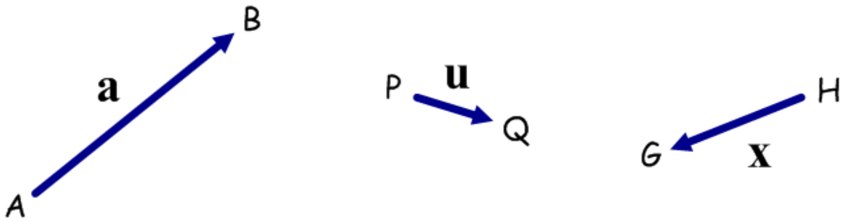


Solution

P is the point (4, 0, 0)
V is the point (0, 2, 3)
Q is the point (4, 2, 0)

Definition of a Vector

A vector is a quantity that has both size and direction. It can be represented as an arrow, where the length of the arrow represents the vector's size (known as a **directed line segment**); and the direction the arrow is pointing in represents its direction.



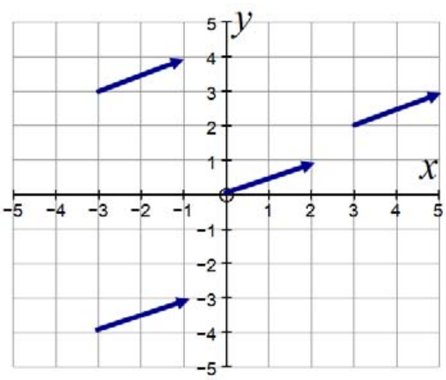
There are two ways of naming a vector:

- One way is to represent a vector by a single letter. For instance in the three examples above, the three vectors are called **a**, **u** and **x**. In print, we use a bold type letter to represent a vector, e.g. **a**. When handwriting, we use underlining in place of bold, e.g. a.
- Another way is to represent a vector using the start and end points. For instance the first vector above goes from A to B, and so it could be represented as $\overrightarrow{AB}$. The middle vector could be represented $\overrightarrow{PQ}$, and the final one would be represented $\overrightarrow{HG}$ (not $\overrightarrow{GH}$).

Components of a vector

A vector is described in terms of its **components**, which describe how far the vector moves in the *x* and *y* directions respectively. For a three-dimensional vector there would be three components, with the third component referring to the *z* direction.

With vectors, the important thing is how the vector moves, not where it begins or starts. All the vectors in the diagram on the right represent the same vector **a**, as both move 2 units in the *x* direction and 1 unit in the *y* direction:



The components of a vector are written in a column. A 2-d vector would be written $\begin{pmatrix} x \\ y \end{pmatrix}$. A 3-d vector would be $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. For example the vector $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ is a 3-d vector moving 1 unit in the x direction, 2 units in the y direction and –3 units in the z direction.

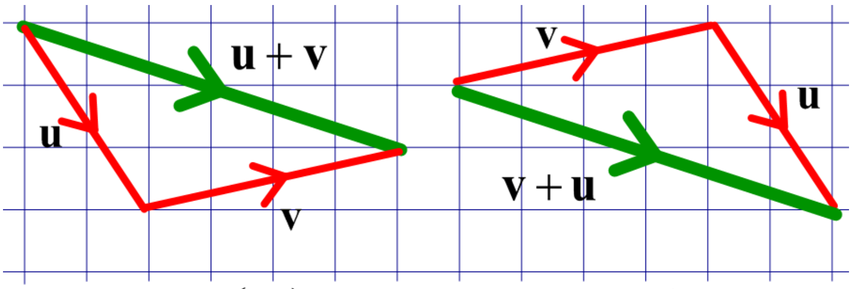
Adding Vectors

We can add vectors to create a **resultant vector**. We can do this in two ways:

- **numerically** by adding their components.

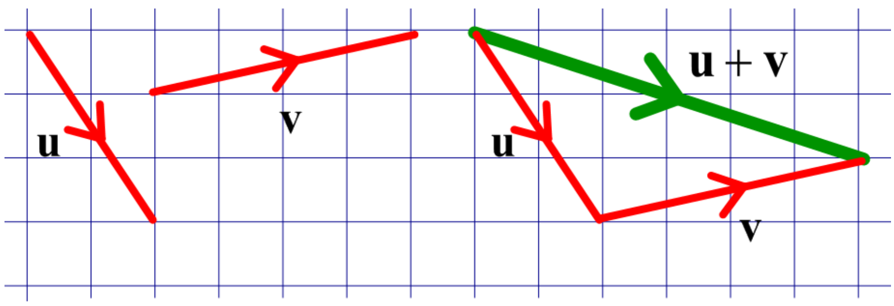
If we have two vectors, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then the resultant vector $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.

- **In a diagram** by joining them 'nose to tail'.



$\mathbf{u} + \mathbf{v} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$ $\mathbf{v} + \mathbf{u} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

It does not matter which order you add vectors in: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.



$\mathbf{u} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ $\mathbf{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ $\mathbf{u} + \mathbf{v} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$

In real life, resultant vectors can be used to work out what the combined effect of more than one force pulling on an object will be.

Example 1 – numerical

Three forces act on an object. The three forces are represented by the vectors a, b and c, where:

$$\mathbf{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0 \\ -5 \\ 6 \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}$$

Find the resultant force.

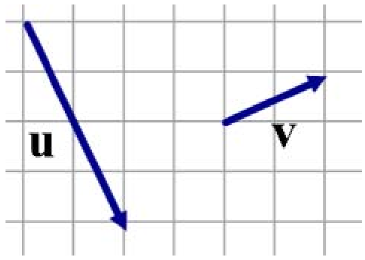
Solution

The resultant force is given b y $\mathbf{a} + \mathbf{b} + \mathbf{c}$.

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} (-1) + 0 + 4 \\ 3 + (-5) + 0 \\ 2 + 6 + 2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 10 \end{pmatrix}$$

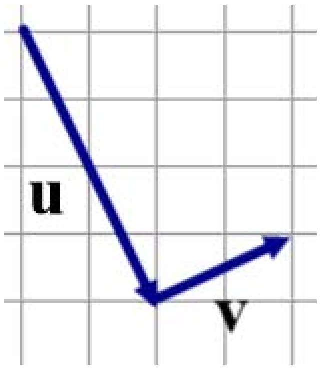
Example 2 – from a diagram

The diagram on the right shows two directed line segments u and v. Draw the resultant vector u + v

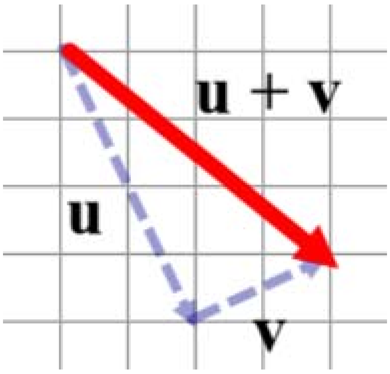


Solution

Too add the vectors, we join the 'tail' of **v** to the 'nose' (pointed end) of **u**:



We can now draw in the vector **u** + **v** going from the 'tail' of **u** to the 'nose' of **v**.



Vector Pathways

We can use the rules of adding and taking away vectors to express a vector $\overrightarrow{AB}$ in a diagram as a combination of other, known, vectors.

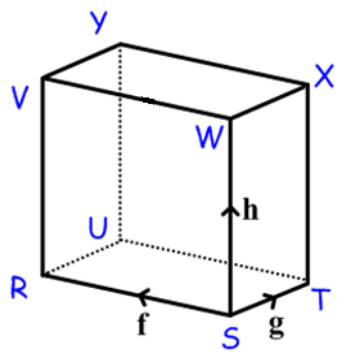
To do this, we identify a route, or pathway, between A and B, in which each step of the route can be expressed in terms of one of the other known pathways. We can choose *any* route we like, and the final answer, when simplified, will always be the same.

Fact: If we move backwards along a vector, we take that vector away.

Example 3 – vector pathways

The diagram shows a cuboid. $\overrightarrow{SR}$ represents vector f, $\overrightarrow{ST}$ represents vector g and $\overrightarrow{SW}$ represents vector h.

Express $\overrightarrow{SU}$ and $\overrightarrow{TV}$ in terms of f, g and h.



Solution

For $\overrightarrow{SU}$:

Step one – identify a pathway from S to U.

One possible pathway is $\overrightarrow{SR}, \overrightarrow{RU}$

Step two – express each part of the pathway in terms of a known vector

$\overrightarrow{SR} = \mathbf{f}$, $\overrightarrow{RU} = \mathbf{g}$

Therefore $\overrightarrow{SU} = \mathbf{f} + \mathbf{g}$

For $\overrightarrow{TV}$:

Step one – identify a pathway from T to V.

One possible pathway is $\overrightarrow{TS}, \overrightarrow{SW}, \overrightarrow{WV}$

Step two – express each part of the pathway in terms of a known vector

$\overrightarrow{TS} =$ backwards along **g**, $\overrightarrow{SW} = \mathbf{h}$, $\overrightarrow{WV} = \mathbf{f}$

Therefore $\overrightarrow{TV} = -\mathbf{g} + \mathbf{h} + \mathbf{f}$ (or $\mathbf{f} - \mathbf{g} + \mathbf{h}$ or any other equivalent expression)

Multiplying a vector by a scalar

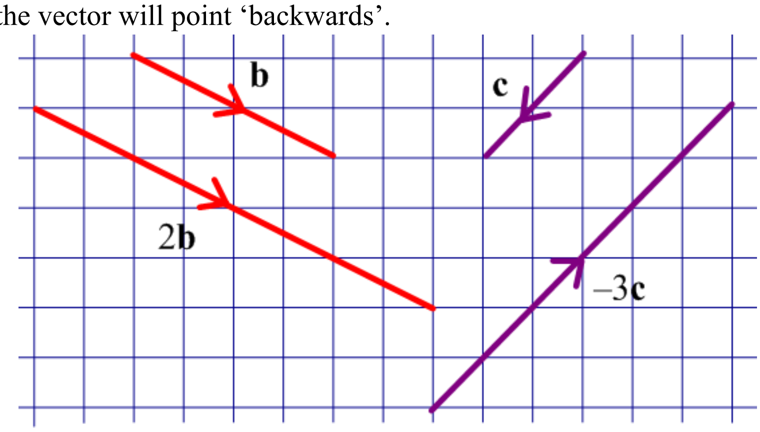
A **scalar** is a quantity that has size but no direction. 'Normal' numbers such as 2, –5 or 14.1 are scalars.

We can multiply a vector by a scalar in two ways:

- **numerically** by multiplying each component of the vector.

If we have a vectors, $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and a scalar k, then $k\mathbf{a} = \begin{pmatrix} ka_1 \\ ka_2 \end{pmatrix}$.

- **In a diagram** by making a vector shorter or longer by a scale factor of k. The vector will still point in the same direction, but will be k times longer (or shorter if $k < 1$). If k is negative, the vector will point 'backwards'.



$\mathbf{b} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ $2\mathbf{b} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$ $\mathbf{c} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$ $-3\mathbf{c} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$

Example

Given that $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$**,** $\mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ **and** $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$**, calculate** $3\mathbf{a} - 2\mathbf{b} + 4\mathbf{c}$

Solution

(a)
$$\begin{aligned} 3\mathbf{a} - 2\mathbf{b} + 4\mathbf{c} &= 3\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} - 2\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 6 \\ -12 \end{pmatrix} - \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -8 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 9-10+4 \\ 6-0+(-8) \\ (-12)-2+12 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \end{aligned}$$

Magnitude

The **magnitude** of a vector is the length of a vector. The magnitude of the vector **a** is written using two vertical lines, $|\mathbf{a}|$.

The magnitude of a two-dimensional vector is found using a version of Pythagoras' Theorem:

Formula. This formula is <u>not given</u> on the National 5 exam paper.

The magnitude of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$

There is also a three-dimensional equivalent of Pythagoras' theorem that can be used to find the magnitude of a 3-d vector when its components are known.

The magnitude of the vector $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

Example 1 – magnitude

Calculate the magnitude of the vector $\mathbf{x} = \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$

Solution

$$\begin{aligned} |\mathbf{x}| &= \sqrt{2^2 + (-5)^2 + 1^2} \\ &= \sqrt{4 + 25 + 1} \\ &= \sqrt{30} \text{ units} \end{aligned}$$

Example 2

Given that $\mathbf{a} = \begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ **and** $\mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, **calculate** $|2\mathbf{a} - 3\mathbf{c}|$

Solution

$$\begin{aligned} 2\mathbf{a} - 3\mathbf{c} &= 2\begin{pmatrix} 3 \\ 2 \\ -4 \end{pmatrix} - 3\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 6 \\ 4 \\ -8 \end{pmatrix} - \begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 10 \\ -17 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{so } |2\mathbf{a} - 3\mathbf{c}| &= \sqrt{3^2 + 10^2 + (-17)^2} \\ &= \sqrt{9 + 100 + 289} \\ &= \sqrt{398} \end{aligned}$$

Percentages

Percentages

In National 5 exam percentage questions, you will always be asked to increase or decrease an amount by a percentage – this will usually be either **compound interest**, or **appreciation** or **depreciation**.

For every question, there is a longer way and a quicker way to do it. Use the one you are happiest with. In the examples below, the quicker method will be preferred.

Percentage	Longer method	Quicker method
3% increase	Multiply by **0.03**, then add answer on	*[100% + 3% = 103%]* Multiply by **1.03**
3% decrease	Multiply by **0.03**, then take answer away	*[100% - 3% = 97%]* Multiply by **0.97**
2.4% increase	Multiply by **0.024**, then add answer on	*[100% + 2.4% = 102.4%]* Multiply by **1.024**
15% decrease	Multiply by **0.15**, then take answer away	*[100% - 15% = 85%]* Multiply by **0.85**
4.5% decrease	Multiply by **0.045**, then take answer away	*[100% - 4.5% = 95.5%]* Multiply by **0.955**

Appreciation and Depreciation

Definition: Appreciation means an increase in value.
Definition: Depreciation means a decrease in value.

Example 1

Peterhead has a population of 30 000. Its population depreciates by 15% per year. What is its population after two years? Round your answer to 3 significant figures.

Solution

Depreciation means decrease, so we will be taking away.
100% – 15% = 85%, so we use **0.85** in the quicker method.
The question is for two years so need to repeat 2 times (a power of 2).

Longer method	Quicker method
Year 1: $0.15 \times 30000 = 4500$ $30000 - 4500 = 25500$	
Year 2: $0.15 \times 25500 = 3825$ $25500 - 3825 = 21675$	$30000 \times 0.85^2 = 21675$
Answer: 21700 (3 s.f.)	**Answer:** 21700 (3 s.f.)

Formula. This formula is **not** given on the National 5 exam paper.

The percentage increase or decrease is found using $\dfrac{\text{increase (or decrease)}}{\text{original amount}} \times 100$

Example 2 – find the percentage first

A house cost £240 000 when first bought. One year later its value has appreciated to £250 800.

a) Find the rate of appreciation (in an exam, this part would have 1 mark)

b) If the house continues to appreciate at this rate, what will its value be after a further 4 years?

Solution

a) The increase is 250 800 – 240 000=£10 800

Using the formula, the percentage increase is given by:

$$\frac{10800}{240000} \times 100 = \underline{\underline{4.5\%}}$$

b) Using the quicker method:

Appreciation means increase, so we will be adding.
100% + 4.5% = 104.5%, so we use **1.045** in the quicker method.
The question is for four years so need to repeat 4 times (a power of 4).
(Note the question says a further *four years – so we start with £250 800 not £240 000).*

$$250800 \times 1.045^4 = 299083.665$$

Answer: after a further 4 years, the house will be worth £299 000 (3 s.f.)

Example 3 – no start value

A vintage car depreciated in value by 5% and then appreciated in value by 12%. How much had its value changed overall?

Solution

It does not matter that we do not have a start value. We can just do the multiplication calculation with the multipliers alone.

The multiplier for a 5% depreciation is 0.95 (since 100 – 5 = 95)
The multiplier for a 12% appreciation is 1.12 (since 100 + 12 = 112)

$$0.95 \times 1.12 = 1.064$$

1.064 is the multiplier for a 6.4% appreciation.

Answer: a 5% depreciation, followed by a 12% depreciation is equivalent to a 6.4% appreciation overall.

Tip: Always use multipliers and powers in any National 5 percentages exam question.

Compound Interest

Compound interest is always an example of **appreciation** (because the amount in the account is always going up), so you always add the amount on each time.

There are two types of questions you may be asked about compound interest:

- If the question asks **how much money** is in the account (the **balance**) you do the appreciation calculation as normal. See Example 1.

- If the question asks what the **interest** was, you do the appreciation calculation and then *subtract* the original amount. See Example 2.

Example 1

A bank account pays 7% interest per annum. £3000 is invested in the account. How much money is in the account after 15 years?

Solution

The longer method is not a good method here unless you want to do 15 lines of calculations. We use the quicker method.

Interest is appreciation. For 7% interest, the multiplier is 1.07.

$$3000 \times 1.07^{15} = 8277.094622$$
$$= \underline{\underline{£8277.09}} \quad \text{(must show units (£) and round to 2 d.p. for money)}$$

Example 2

A savings account pays 2.4% interest per annum. If you put £2500 in the account, how much compound interest will you receive after 3 years?

Longer method	Quicker method
Year 1: 0.024 × 2500 = £60 2500 + 60 = £2560 **Year 2:** 0.024 × 2560 = £61.44 2560 + 61.44 = £2621.44 **Year 3:** 0.024 × 2621.44 = £62.91 2621.44 + 62.91 = £2684.35	$2500 \times 1.024^3 = 2684.35$
Interest: 2684.35 – 2500 = £184.35	**Interest:** 2684.35 – 2500 = £184.35

Reversing a Percentage Change

To go backwards with a percentage change, we divide by the multiplier instead multiplying.

Example 1

Andy gets a 3% pay rise. After the pay rise he is earning £40 685 a year. What was he earning each year before the pay rise?

Solution

For a 3% increase, the multiplier is 1.03.

40 685 ÷ 1.03 = £39 500

Example 2

A ship's value depreciates by 2.5% a year. After two years, it is worth £240 000. How much was it worth originally? Round your answer to two significant figures.

Solution

For a 2.5% depreciation, the multiplier is 0.975. For two years, we use a power of 2.

$240\,000 \div 0.975^2 = £252\,465 = £250\,000$ (2s.f.)

Fractions and Mixed Numbers

Topheavy Fractions and Mixed Numbers

In an exam, you would be expected to add, take away, multiply or divide fractions, including topheavy fractions (e.g. $\frac{7}{2}$) or those which may be expressed as a mixed number (e.g. $3\frac{1}{2}$).

You need to be able to change a fraction from a topheavy fraction or vice versa. To do this we can think about the link between fractions and dividing.

$\frac{a}{b} = a \div b = p\frac{q}{b}$, where p is the quotient and q is the remainder when doing $a \div b$.

Example 1

Change the topheavy fractions to mixed numbers: (a) $\frac{13}{3}$ (b) $\frac{46}{7}$

Solutions

(a) $13 \div 3 = \mathbf{4}\text{ r }\mathbf{1}$, so $\frac{13}{3} = 4\frac{1}{3}$ (b) $46 \div 7 = \mathbf{6}\text{ r }\mathbf{4}$, so $\frac{46}{7} = 6\frac{4}{7}$

To go the other way, we use the fact that $a\frac{b}{c} = \frac{ac+b}{c}$

Example 2

Change the mixed numbers to topheavy fractions: (a) $6\frac{2}{5}$ (b) $10\frac{4}{9}$

Solutions

(a) $5 \times 6 + 2 = \mathbf{32}$, so $6\frac{2}{5} = \frac{32}{5}$ (b) $9 \times 10 + 4 = \mathbf{94}$, so $10\frac{4}{9} = \frac{94}{9}$

Adding and Subtracting Fractions

You can only add and subtract fractions when the denominators are the same. When they are not the same, we have to change the fractions into another fraction that *is* the same.

A quick method for doing this, and the one used in these notes, is known as the **'kiss and smile'** method because of the shape formed when you draw lines between the terms you are combining. The method is outlined and explained on page 19.

Example 1

Take away: $\frac{7}{8} - \frac{2}{3}$

Solution

Step one (smile) $\frac{7}{8} - \frac{2}{3}$

Step two (kiss) $\frac{7}{8}\ \frac{2}{3}$

Step three – take away (or add) the top line

$$\frac{7}{8} - \frac{2}{3} = \frac{\quad}{24}$$

$$= \frac{21-16}{24}$$

$$= \frac{5}{24}$$

Adding works in exactly the same way as taking away. The only difference is that step three involves an add sum rather than a take away sum.

If either or both the numbers are mixed numbers, then the 'whole number' part and the 'fraction' part can be dealt with separately.

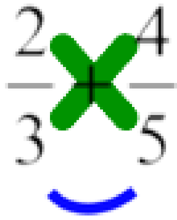
Example 2

Add: $\mathbf{4\frac{2}{3}+3\frac{4}{5}}$

Solution

To add the whole numbers we just do 4 + 3 = 7

Now we use 'kiss and smile' to add $\frac{2}{3}$ and $\frac{4}{5}$.



$$\begin{aligned}\frac{2}{3}+\frac{4}{5}&=\frac{10+12}{15}\\&=\frac{22}{15}\\&=1\frac{7}{15}\end{aligned}$$

Finally add 7 and $1\frac{7}{15}$. The final answer is $8\frac{7}{15}$.

Multiplying and Dividing Fractions

We multiply and divide fractions in the same way as we do for algebraic fractions (shown on page 18).

Multiplying fractions is a straightforward procedure – you **multiply the tops and multiply the bottoms**.

e.g. $\frac{3}{5}\times\frac{2}{7}=\frac{3\times 2}{5\times 7}=\frac{6}{35}$ $\qquad\frac{a}{c}\times\frac{b}{c}=\frac{ab}{c^2}$

It is easiest to cancel before you multiply. You are allowed to cancel *anything* from the top row with *anything* from the bottom row.

Example 1 – multiplying with cancelling

Write a single fraction in its simplest form: $\mathbf{\frac{35}{34}\times\frac{17}{42}}$

Solution

Cancelling gives: .

$$\begin{aligned}\frac{35}{34}\times\frac{17}{42}&=\frac{\overset{5}{\cancel{35}}}{34}\times\frac{17}{\underset{6}{\cancel{42}}} &&\text{(cancelling factors of 7)}\\&=\frac{5}{\cancel{34}_2}\times\frac{\overset{1}{\cancel{17}}}{6} &&\text{(cancelling factors of 17)}\\&=\frac{5}{2}\times\frac{1}{6}\\&=\underline{\underline{\frac{5}{12}}} &&\text{(multiplying tops and bottoms)}\end{aligned}$$

To multiply mixed numbers, we have to change them to topheavy fractions first.

Example 2 – multiplying mixed numbers

Multiply $\mathbf{2\frac{1}{4}\times 1\frac{3}{5}}$

Solution

First change to topheavy fractions. $2\frac{1}{4}\times 1\frac{3}{5}$ becomes $\frac{9}{4}\times\frac{8}{5}$. Next:

$$\frac{9}{4}\times\frac{8}{5}=\frac{9}{{}_1\cancel{4}}\times\frac{\cancel{8}^2}{5} \qquad \text{(cancelling a factor of 4)}$$

$$=\frac{9}{1}\times\frac{2}{5}$$

$$=\frac{18}{5} \quad \left(\text{or } 3\frac{3}{5}\right)$$

To divide two fractions, you:

1. flip the second fraction upside down
2. and change the sum to be a multiply sum:

e.g. $\frac{3}{5}\div\frac{2}{7}=\frac{3}{5}\times\frac{7}{2}=\frac{21}{10}$ $\qquad$ $\frac{x}{y}\div\frac{a}{x}=\frac{x}{y}\times\frac{x}{a}=\frac{x^2}{ay}$

Example 3 – dividing mixed numbers

Divide: $\mathbf{4\frac{1}{3}\div\frac{4}{5}}$

Solution

First change to topheavy fractions. $4\frac{1}{3}\div\frac{4}{5}$ becomes $\frac{13}{3}\div\frac{4}{5}$. Next:

$$\frac{13}{3}\div\frac{4}{5}=\frac{13}{3}\times\frac{5}{4} \qquad \text{(flipping and multiplying)}$$

$$=\frac{13\times 5}{3\times 4}$$

$$=\frac{65}{12} \quad \left(\text{or } 5\frac{5}{12}\right)$$

Statistics

Scatter Graphs and Line of Best Fit

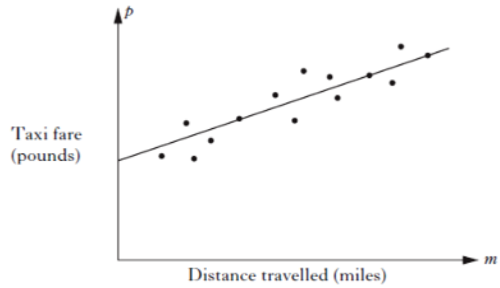
The line of best fit on a scattergraph is a straight line. This means that you can find the equation of a line of best fit using the method ($y = mx + c$) on page 27.

Once you have the equation, you can use the equation to estimate the value of y when you are told x (or vice versa). At National 5, **you have to use the equation** to get any marks (the question will say this). You *cannot* do it by "looking and guessing". Any answer without working will get zero marks, even if it happens to be correct.

Example 1 (2010 Intermediate 2 Exam Question)

A scattergraph shows the taxi fare *p* pounds plotted against the distance travelled, *m* miles. A line of best fit has been drawn.

The equation of the line of best fit is $p = 2 + 1.5m$. Use this equation to predict the taxi fare for a journey of 6 miles.



Solution

The journey is 6 miles, so $m = 6$. Using the equation,

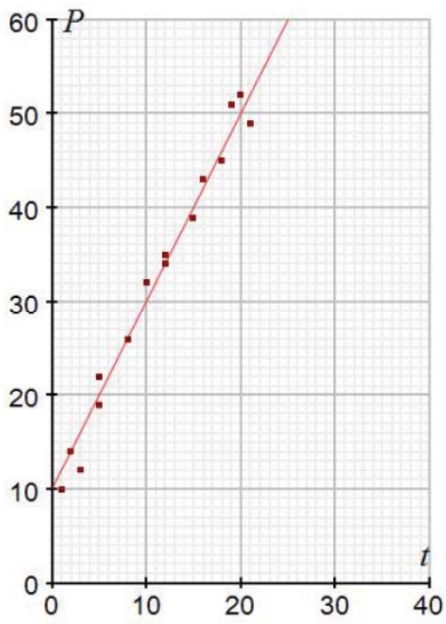
$$p = 2 + 1.5m$$
$$p = 2 + 1.5 \times 6$$
$$p = \underline{11 \text{ miles}}$$

Example 2

The scattergraph shows the power of an industrial battery (*P*) after *t* hours of charging.

(a) Find the equation of the line of best fit

(b) Use your equation to estimate the power of a battery that has been charged for 60 hours.



Solution

(a) We use the usual method for $y = mx + c$ (see page 29). On this occasion, since the letters P and t are being used, we will use $P = mt + c$.

The y intercept is 10, so $c = 10$.

Now choose two points on the line (not necessarily on the original scatter graph) to calculate the gradient.

Two points on the line are (0, 10) and (10, 30). Therefore the gradient is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{30 - 10}{10 - 0} = \frac{20}{10}$$
$$= 2$$

The line of best fit has y-intercept 10 and gradient 2, so the equation is $y = 2x + 10$, or (using the correct letters for this question) $P = 2t + 10$.

Note we could also have used $y - b = m(x - a)$ ***to obtain the same answer.***

(b) We use our equation from part (a) and put the number from the question into it.

The question uses the number 60, so we use $t = 60$.

$$\begin{aligned} P &= 2t + 10 \\ &= 2 \times 60 + 10 \\ &= 130 \end{aligned}$$

Median, Quartiles and Semi Interquartile Range

Definition: the **median** is the number that divides an ordered list of numbers into two equally-sized parts
Definition: the **quartiles**, along with the median, are the numbers that divide an ordered list into four equally-sized parts. The list must be written in order.

Formula. This formula **is not** given on the National 5 exam paper.
Semi Interquartile-Range: $\text{SIQR} = \dfrac{\text{upper quartile} - \text{lower quartile}}{2}$

Example

For the following set of numbers, calculate the median, quartiles and semi inter-quartile range:

12, 14 , 15, 15, 16, 17, 17, 19, 19, 20, 21, 22, 26, 28, 31, 31

Solution

First write the list out in order and draw arrows to identify the median and quartiles.

L lower Q Median upper Q H

12, 14 , 15, 15, | 16, 17, 17, 19, | 19, 20, 21, 22, | 26, 28, 31, 31

15.5 19 24

Lowest value	L= 12
Lower Quartile	$Q_1 = 15.5$
Median	$Q_2 = 19$
Upper Quartile	$Q_3 = 24$
Highest value	H = 31

Upper quartile = 24 and Lower quartile = 15.5, so $\text{SIQR} = \dfrac{24 - 15.5}{2} = \underline{4.25}$

Standard Deviation

Definition: the standard deviation of a list of numbers is a measure of how spread out the numbers are from the mean.

Formula. These formulae are <u>given</u> on the National 5 exam paper.

$$\text{standard deviation} = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}} \quad \text{or} \quad \sqrt{\frac{\sum x^2 - \left(\sum x\right)^2 / n}{n-1}}$$

Where n is how many numbers are in the list, $\bar{x}$ is the mean and Σ means "add together"

You only need to use one of these formulae. In general, it is more helpful to just know the method rather than memorising the formula.

<u>Example</u>

a) Find the mean of these five numbers: 2, 3, 9, 6, 5
b) Find the standard deviation of the same five numbers

Solution

a) $\frac{2+3+9+6+5}{5} = \frac{25}{5} = 5$, so the <u>mean is 5</u>

b) You have a choice of two methods:

Method 1 – using the formula $s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$

<u>Step 1</u> - Draw up a table showing x, $x-\bar{x}$ and $(x-\bar{x})^2$

x	$x-\bar{x}$	$(x-\bar{x})^2$
2		
3		
9		
6		
5		

<u>Step 2</u> – Complete the table, remembering that $\bar{x}$ = the mean = 5.

<u>Step 3</u> – find the total of the final column

So $\sum(x-\bar{x})^2 = 30$

	x	$x-\bar{x}$	$(x-\bar{x})^2$
	2	–3	9
	3	–2	4
	9	4	16
	6	1	1
	5	0	0
TOTAL			**30**

<u>Step 4</u> – use the formula, remembering that $n = 5$ as there were five numbers.

$$s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$$

$$= \sqrt{\frac{30}{5-1}}$$

$$= \sqrt{\frac{30}{4}} = 2.74 \text{ (2 d.p.)}$$

Method 2 – using the formula: $s=\sqrt{\dfrac{\sum x^2-\left(\sum x\right)^2/n}{n-1}}$

Step 1 - Draw up a table showing x and x^2

x	x^2
2	
3	
9	
6	
5	

Step 2 – Complete the table

Step 3 – find the totals

So $\sum x=25,\ \sum x^2=155$

	x	x^2
	2	4
	3	9
	9	81
	6	36
	5	25
TOTAL	**25**	**155**

Step 4 – use the formula, remembering that n = 5 as there are 5 numbers.

$$s=\sqrt{\frac{\sum x^2-\left(\sum x\right)^2/n}{n-1}}$$

$$=\sqrt{\frac{155-25^2/5}{5-1}}=\sqrt{\frac{(155-125)}{4}}$$

$$=\sqrt{\frac{30}{4}}$$

$$=2.74\text{ (2 d.p.)}$$

Comparing Statistics

The **mean**, **median** and **mode** are averages. They say whether a list of numbers is higher or lower on average.

The **range**, **semi interquartile range** and **standard deviation** are measures of spread. They say whether a list of numbers is more or less spread out/varied/consistent.

- A lower range, IQR or standard deviation means the numbers are more **consistent**.
- A higher range, IQR or standard deviation means the numbers are more **varied**.

Example

The temperature in Aberdeen has a mean of 3°C and a standard deviation of 5.
In London it has a mean of 9°C and a standard deviation of 3.
Compare the temperatures in London and Aberdeen.

Solution

You would get NO MARKS (as you are stating the obvious) for:
"Aberdeen has a lower mean", "London has a higher mean", "Aberdeen has a higher standard deviation", "London has a lower standard deviation".
You WOULD get marks (as you say what the numbers MEAN) for:
"The temperature in Aberdeen is lower and the temperature is less consistent"
"The temperature in London is higher and more consistent" or similar

Index of Key Words

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