Differentiation

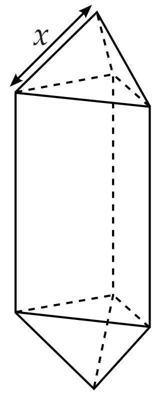
[SQA] 1. A goldsmith has built up a solid which consists of a triangular prism of fixed volume with a regular tetrahedron at each end.

The surface area, A, of the solid is given by

$$A(x) = \frac{3\sqrt{3}}{2}\left(x^2 + \frac{16}{x}\right)$$

where x is the length of each edge of the tetrahedron.

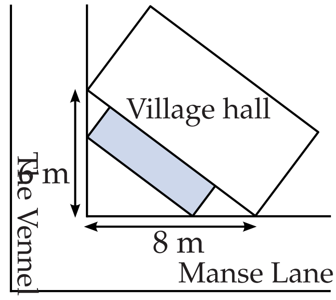
Find the value of x which the goldsmith should use to minimise the amount of gold plating required to cover the solid. **6**



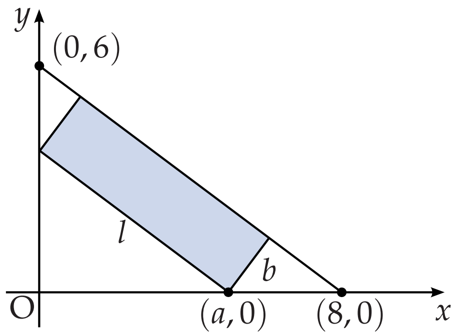
Part	Marks	Level	Calc.	Content	Answer	U1 OC3
	6	A/B	CN	C11	$x = 2$	2000 P2 Q6

•1 ss: know to differentiate •2 pd: process •3 ss: know to set $f'(x) = 0$ •4 pd: deal with x^{-2} •5 pd: process •6 ic: check for minimum	•1 $A'(x) = \ldots$ •2 $\frac{3\sqrt{3}}{2}(2x - 16x^{-2})$ **or** $3\sqrt{3}x - 24\sqrt{3}x^{-2}$ •3 $A'(x) = 0$ •4 $-\frac{16}{x^2}$ **or** $-\frac{24\sqrt{3}}{x^2}$ •5 $x = 2$ •6 x: 2^-, 2, 2^+; $A'(x)$: $-ve$, 0, $+ve$ so $x = 2$ is min.

[SQA] 2. The shaded rectangle on this map represents the planned extension to the village hall. It is hoped to provide the largest possible area for the extension.



The coordinate diagram represents the right angled triangle of ground behind the hall. The extension has length l metres and breadth b metres, as shown. One corner of the extension is at the point $(a, 0)$.



(*a*) (i) Show that $l = \frac{5}{4}a$.

(ii) Express b in terms of a and hence deduce that the area, A m^2, of the extension is given by $A = \frac{3}{4}a(8 - a)$. **3**

(*b*) Find the value of a which produces the largest area of the extension. **4**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	3	A/B	CN	CGD	proof	2002 P2 Q10
(*b*)	4	A/B	CN	C11	$a = 4$	

•1 ss: select strategy and carry through •2 ss: select strategy and carry through •3 ic: complete proof •4 ss: know to set derivative to zero •5 pd: differentiate •6 pd: solve equation •7 ic: justify maximum, e.g. nature table	•1 proof of $l = \frac{5}{4}a$ •2 $b = \frac{3}{5}(8 - a)$ •3 complete proof leading to $A = \ldots$ •4 $\frac{dA}{da} = \ldots = 0$ •5 $6 - \frac{3}{2}a$ •6 $a = 4$ •7 e.g. nature table, comp. the square

[SQA] 3. Diagram 1 is an artist's impression of a new warehouse based on the architect's plans. The warehouse is in the shape of a cuboid and is supported by three identical parabolic girders spaced 30 metres apart.

With coordinate axes as shown in Diagram 2, the shape of each girder can be described by the equation $y = 9 - \frac{1}{4}x^2$.

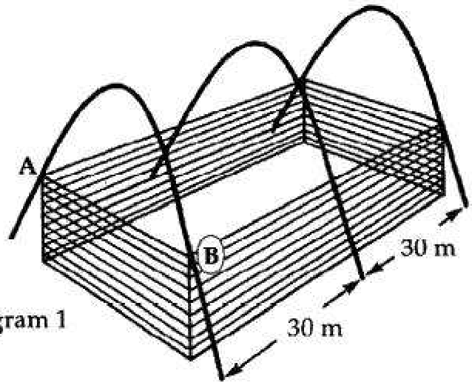


Diagram 1

(*a*) Given that AB is $2x$ metres long, show that the shaded area in Diagram 2 is $\left(18x - \frac{1}{2}x^2\right)$ square metres. **(2)**

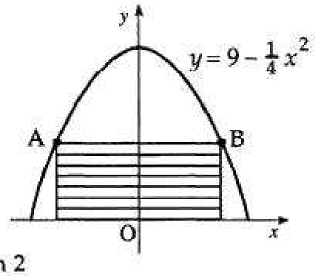


Diagram 2

(*b*) The architect wished to fit into the girders the cuboidal warehouse which had the maximum volume. Find the value of this maximum volume. **(6)**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	2	C	CN	A6		1989 P2 Q7
(*b*)	3	C	CN	C11		
(*b*)	3	A/B	CN	C11		

(*a*) •[1] $B = (x, y)$ where $y = 9 - \frac{1}{4}x^2$

•[2] area $= 2x\left(9 - \frac{1}{4}x^2\right)$

(*b*) •[3] $V = 1080x - 30x^3$

•[4] $\frac{dV}{dx} = 1080 - 90x^2$

•[5] $\frac{dV}{dx} = 0$ stated explicitly

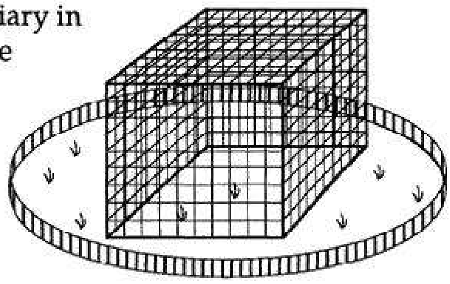
•[6] $x = 2\sqrt{3}$

•[7]

x	$2\sqrt{3}^-$	$2\sqrt{3}$	$2\sqrt{3}^+$
$\frac{dV}{dx}$	$+$	0	$-$

•[8] max at $x = 2\sqrt{3}$ of $1440\sqrt{3}$

[SQA] 4. The owners of a zoo intend to build a new aviary in the shape of a cuboid with a square floor. The volume of the aviary will be 500 m^3.



(*a*) If x metres is the length of one edge of the floor, show that the area A square metres of netting required is given by

$$A = x^2 + \frac{2000}{x}.$$

(4)

(*b*) Find the dimensions of the aviary to ensure that the cost of netting is minimised. **(6)**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	2	C	NC	A6		1992 P2 Q5
(*a*)	2	A/B	NC	A6		
(*b*)	4	C	NC	C11		
(*b*)	2	A/B	NC	C11		

(a) •1 introduce height specific to this cuboid

•2 $h = \frac{500}{x^2}$

•3 $A = x^2 + 4xh$

•4 $A = x^2 + 4x.\frac{500}{x^2}$ explicitly stated

(b) •5 $A'(x) = \ldots\ldots$

•6 $2x - 2000x^{-2}$

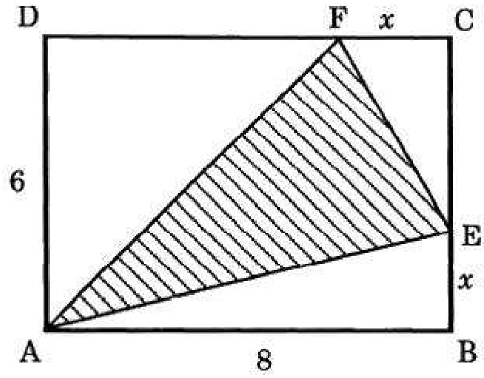
•7 $A'(x) = 0$ specifically stated

•8 $x = 10$

•9 justify minimum e.g. with table

•10 dimensions of 10 by 10 by 5

[SQA] 5. An yacht club is designing its new flag. The flag is to consist of a red triangle on a yellow rectangular background. In the yellow rectangle ABCD, AB measures 8 units and AD is 6 units. E and F lie on BC and CD, x units from B and C as shown in the diagram.



(*a*) Show that the area, H square units, of the red triangle AEF is given by $H(x) = 24 - 4x + \frac{1}{2}x^2$. **(4)**

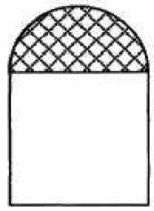
(*b*) Hence find the greatest and least possible values of the area of triangle AEF. **(8)**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	4	C	NC	CGD		1994 P2 Q7
(*b*)	3	C	NC	C11		
(*b*)	5	A/B	NC	C11		

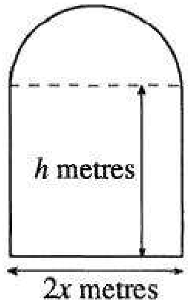
(*a*)
- •[1] rectangle minus 3 triangles
- •[2] area of Δ's ADF and ABE
- •[3] area of Δ FCE
- •[4] 3 triangles : $24 + 4x - \frac{1}{2}x^2$ or $48 - 4x - 3x + \frac{1}{2}x^2 - 24 + 3x$

(*b*)
- •[5] $H'(x) = \ldots\ldots$
- •[6] $x - 4$
- •[7] put $H'(x) = 0$ stated explicitly
- •[8] $x = 4$ and $H = 16$
- •[9] justify minimum
- •[10] consider $x = 0$ and $x = 6$
- •[11] $H(0) = 24$, and $H(6) = 18$
- •[12] communication re greatest and least.

[SQA] 6. A window in the shape of a rectangle surmounted by a semicircle is being designed to let in the maximum amount of light.
The glass to be used for the semicircular part is stained glass which lets in one unit of light per square metre; the rectangular part uses clear glass which lets in 2 units of light per square metre.



The rectangle measures $2x$ metres by h metres.



(*a*) (i) If the perimeter of the whole window is 10 metres, express h in terms of x. **(2)**

(ii) Hence show that the amount of light, L, let in by the window is given by $L = 20x - 4x^2 - \frac{3}{2}\pi x^2$. **(2)**

(*b*) Find the values of x and h that must be used to allow this design to let in the maximum amount of light. **(5)**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	1	C	CN	CGD		1996 P2 Q11
(*a*)	3	A/B	CN	CGD		
(*b*)	2	C	CN	C11		
(*b*)	3	A/B	CN	C11		

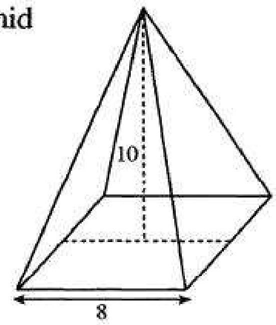
(a)
•[1] eg $2h + 2x + \text{semicircle} = 10$
•[2] $h = \frac{1}{2}(10 - \pi x - 2x)$
•[3] $L = 2 \times 2xh + \frac{1}{2}\pi x^2$
•[4] $L = 4x \times \frac{1}{2}(10 - \pi x - 2x) + \frac{1}{2}\pi x^2$
$L = 20x - 4x^2 - \frac{3}{2}\pi x^2$

(b)
•[5] $L' = 20 - 8x - 3\pi x$
•[6] $L' = 0$
•[7] $x = \frac{20}{3\pi + 8} = x_0$ $(= 1.148)$
•[8]

x	x_0^-	x_0	x_0^+
L'	$+$	0	$-$
maximum at x_0			

•[9] $h = \frac{5\pi + 20}{3\pi + 8}$ $(= 2.049)$

[SQA] 7. A cuboid is to be cut out of a right square-based pyramid. The pyramid has a square base of side 8 cm. and a vertical height of 10cm.

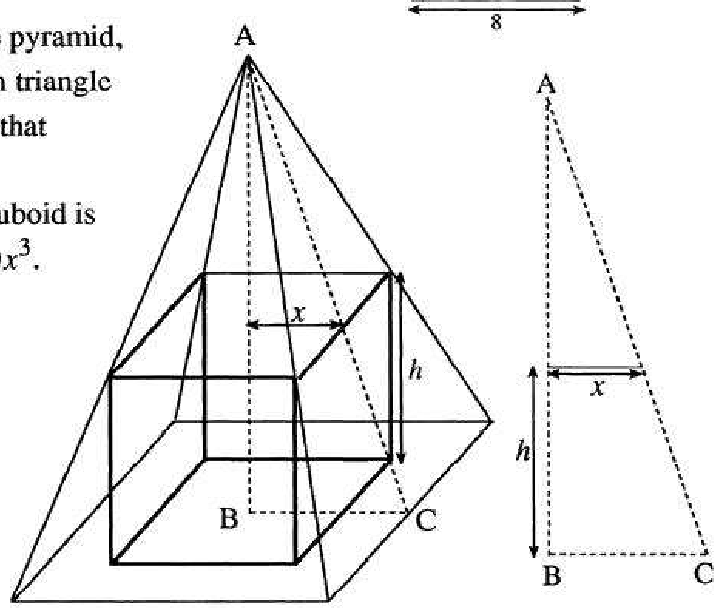


(*a*) The cuboid has a square base of side $2x$ cm and a height of h cm.

If the cuboid is to fit into the pyramid, use the information shown in triangle ABC, or otherwise, to show that

(i) $h = 10 - \frac{5}{2}x$. **(3)**

(ii) the volume, V, of the cuboid is given by $V = 40x^2 - 10x^3$. **(1)**



(*b*) Hence find the dimensions of the square-based cuboid with the greatest volume which can be cut from the pyramid. **(6)**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	1	C	CN	CGD		1997 P2 Q10
(*a*)	3	A/B	CN	CGD		
(*b*)	3	C	CN	C11		
(*b*)	3	A/B	CN	C11		

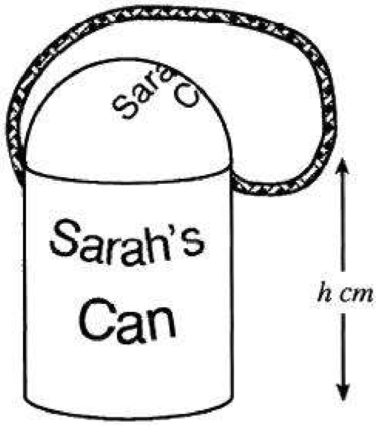
(a)
- •1 strategy: e.g. equate ratios from similar triangles
- •2 $\frac{10}{4} = \frac{10-h}{x}$ or equivalent
- •3 complete proof
- •4 $V = 40x^2 - 10x^3$

(b)
- •5 $\frac{dV}{dx} =$
- •6 $80x - 30x^2$
- •7 $\frac{dV}{dx} = 0$ for stationary points
- •8 $0, \frac{8}{3}$
- •9

x	…	$\frac{8}{3}$	…
$\frac{dV}{dx}$	+	0	−
		max	

- •10 $\frac{16}{3}$ and $\frac{10}{3}$

[SQA] 8. A child's drinking beaker is in the shape of a cylinder with a hemispherical lid and a circular flat base. The radius of the cylinder is r cm and the height is h cm. The volume of the cylinder is 400 cm^3.



(*a*) Show that the surface area of plastic, $A(r)$, needed to make the beaker is given by $A(r) = 3\pi r^2 + \frac{800}{r}$. **(3)**

Note: The curved surface area of a hemisphere of radius r is $2\pi r^2$.

(*b*) Find the value of r which ensures that the surface area of plastic is minimised. **(6)**

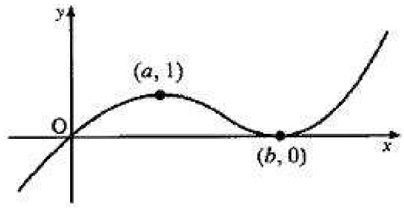
Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	3	A/B	CR	CGD		1998 P2 Q10
(*b*)	3	C	CR	C11		
(*b*)	3	A/B	CR	C11		

(a)
•1 $\pi r^2 + 2\pi rh + 2\pi r^2$
•2 $h = \frac{400}{\pi r^2}$ or equivalent (*e.g.* $\pi rh = \frac{400}{r}$)
•3 $2\pi r \frac{400}{\pi r^2} + 3\pi r^2$ and completes proof

(b)
•4 $\frac{dA}{dr} = \ldots$
•5 $800r^{-1}$
•6 $6\pi r - 800r^{-2}$
•7 *e.g.* $6\pi r - \frac{800}{r^2} = 0$
•8 3.5
•9

r	3.5^-	3.5	3.5^+
$\frac{dA}{dr}$	$-ve$	0	$+ve$

[SQA] 9. A sketch of the graph of the cubic function f is shown. It passes through the origin, has a maximum turning point at $(a, 1)$ and a minimum turning point at $(b, 0)$.



(a) Make a copy of this diagram and on it sketch the graph of $y = 2 - f(x)$, indicating the coordinates of the turning points. 3

(b) On a separate diagram sketch the graph of $y = f'(x)$. 2

(c) The tangent to $y = f(x)$ at the origin has equation $y = \frac{1}{2}x$. Use this information to write down the coordinates of a point on the graph of $y = f'(x)$. 1

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(a)	3	C	CN	A3		1998 P1 Q13
(b)	2	A/B	CN	A3		
(c)	1	A/B	CN	C4		

•1 clear evidence of reflection in $y = 0$

•2 clear evidence of translation $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ subsequent to a reflection

•3 indication of passing through $(a,1)$ and $(b,2)$

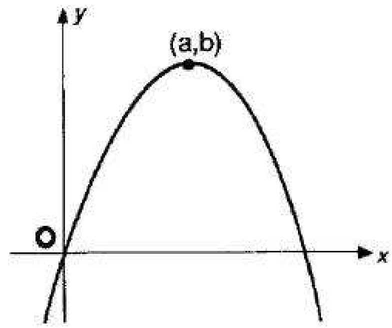
•4 roots at $x = a$ and $x = b$

•5 parabolic shape with min. turning point between the roots and no other turning points

•6 $(0, \frac{1}{2})$

[SQA] 10. The line with equation $y = x$ is a tangent at the origin to the parabola with equation $y = f(x)$. The parabola has a maximum turning point at (a, b).

Sketch the graph of $y = f'(x)$. 4



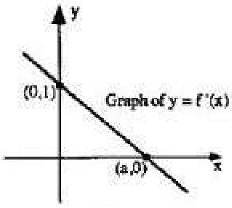
Part	Marks	Level	Calc.	Content	Answer	U1 OC3
	1	C	CN	C4, A3		1992 P1 Q19
	3	A/B	CN	C4, A3		

•1 $f'(a) = 0$

•2 m_{tgt} at $(0,0) = 1$

•3 $f'(0) = 1$

•4 for the sketch

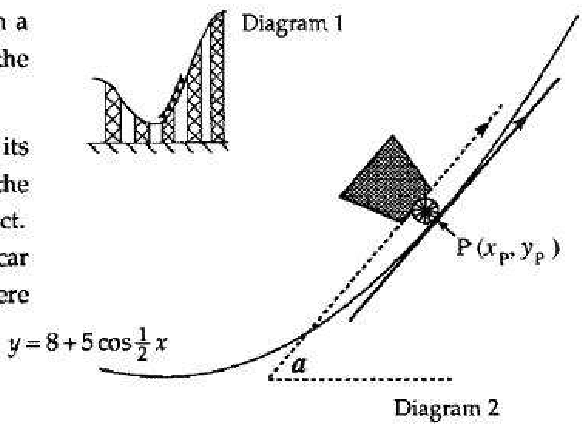


[SQA] 11. Diagram 1 shows 5 cars travelling up an incline on a roller-coaster. Part of the roller-coaster rail follows the curve with equation $y = 8 + 5\cos\frac{1}{2}x$.

Diagram 2 shows an enlargement of the last car and its position relative to a suitable set of axes. The floor of the car lies parallel to the tangent at P, the point of contact.

Calculate the acute angle a between the floor of the car and the horizontal when the car is at the point where $x_P = \frac{7\pi}{3}$.

Express your answer in degrees. **4**



Part	Marks	Level	Calc.	Content	Answer	U1 OC3
	1	C	CR	C4, C4, G2		1997 P1 Q20
	3	A/B	CR	C4, C4, G2		

•1 $\frac{dy}{dx} = \ldots$

•2 $5 \times \left(-\frac{1}{2}\sin\frac{1}{2}x\right)$

•3 $m = \frac{5}{4}$

•4 $\theta = 51.3°$

[SQA] 12. A ball is thrown vertically upwards. The height h metres of the ball t seconds after it is thrown, is given by the formula $h = 20t - 5t^2$.

(*a*) Find the speed of the ball when it is thrown (i.e. the rate of change of height with respect to time of the ball when it is thrown). **3**

(*b*) Find the speed of the ball after 2 seconds.

Explain your answer in terms of the movement of the ball. **2**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	1	C	NC	C6		1995 P1 Q21
(*a*)	2	A/B	NC	C6		
(*b*)	2	A/B	NC	A6		

•1 knows to differentiate

•2 $20 - 10t$

•3 20

•4 speed = 0

•5 ball stationary at top of flight

[SQA] 13. A ball is thrown vertically upwards.

After t seconds its height is h metres, where $h = 1{\cdot}2 + 19{\cdot}6t - 4{\cdot}9t^2$.

(*a*) Find the speed of the ball after 1 second. **3**

(*b*) For how many seconds is the ball travelling upwards? **2**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
(*a*)	1	C	CN	C6, C6		1998 P1 Q17
(*a*)	2	A/B	CN	C6, C6		
(*b*)	2	A/B	CN	C6, C6		

•[1] $\frac{dh}{dt} = \ldots\ldots$

•[2] $19.6 - 9.8t$

•[3] 9.8

•[4] $\frac{dh}{dt} = 0$

•[5] $t = 2$

Alternative

•[4] $h(t)$ is a parabola which is **symmetric** about its **maximum**

•[5] (*e.g.*) $h(1) = 15{\cdot}9,\ h(2) = 20.8,\ h(3) = 15{\cdot}9$ so $t = 2$

[SQA] 14. For what values of x is the function $f(x) = \frac{1}{3}x^3 - 2x^2 - 5x - 4$ increasing? **5**

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
	2	C	NC	C7		1990 P1 Q16
	3	A/B	NC	C7		

•[1] $f'(x) = x^2 - 4x - 5$

•[2] use $f'(x) > 0$

•[3] zeros at $x = 5$ and $x = -1$

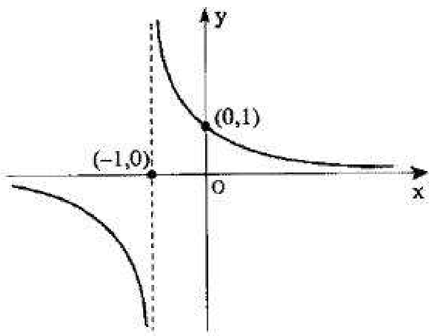
•[4] *strat.* e.g. for $-1 < x < 5$ test $x = 0$

•[5] $x < -1,\ x > 5$

[SQA] 15. The diagram shows the graph of the function

$$f(x) = \frac{1}{x+1}, \quad x \neq -1.$$

Prove that the function f is decreasing for all values of x except $x = -1$.

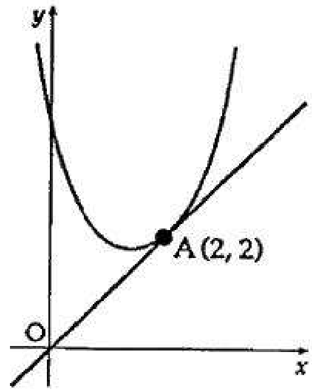


4

Part	Marks	Level	Calc.	Content	Answer	U1 OC3
	1	C	NC	C7		1993 P1 Q21
	3	A/B	NC	C7		

- •[1] show that $f'(x) < 0$
- •[2] $f(x) = (x+1)^{-1}$
- •[3] $f'(x) = \dfrac{-1}{(x+1)^2}$
- •[4] explaining that $(x+1)^2 > 0 \Rightarrow f'(x) < 0$

[SQA] 16. (*a*) The point A(2, 2) lies on the parabola $y = x^2 + px + q$. Find a relationship between p and q. **(1)**



(*b*) The tangent to the parabola at A is the line $y = x$. Find the value of p. Hence find the equation of the parabola. **(6)**

(*c*) Using your answers for p and q, find the value of the discriminant of $x^2 + px + q = 0$. What feature of the above sketch is confirmed by this value? **(2)**

Part	Marks	Level	Calc.	Content	Answer	U2 OC1
(*a*)	1	C	CN	A6		1994 P2 Q9
(*b*)	2	C	CN	C4, CGD		
(*b*)	4	A/B	CN	C4, CGD		
(*c*)	2	A/B	CN	A17		

(*a*) •1 $2p + q = -2$

(*b*) •2 strategy

•3 $2x + p$

•4 gradient = 1, or equivalent

•5 $4 + p$

•6 $p = -3$

•7 $q = 4$

(*c*) •8 $\Delta = -7$

•9 $\sqrt{-7}$ means no roots

[SQA] 17. Find the values of x for which the function $f(x) = 2x^3 - 3x^2 - 36x$ is increasing. **4**

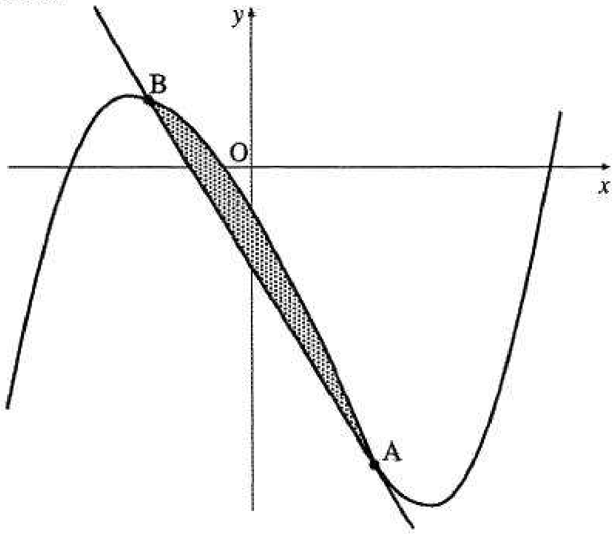
Part	Marks	Level	Calc.	Content	Answer	U2 OC1
	2	C	NC	C7, A16		1996 P1 Q16
	2	A/B	NC	C7, A16		

•¹	know to consider $f'(x) > 0$	stated or implied by the evidence for •⁴.
•²	$\frac{dy}{dx} = 6x^2 - 6x - 36$	
•³	$6(x-3)(x+2) > 0$	or by formula or completing the square
•⁴	$x < -2,\ x > 3$	

[SQA] 18. In the diagram below a winding river has been modelled by the curve $y = x^3 - x^2 - 6x - 2$ and a road has been modelled by the straight line AB. The road is a tangent to the river at the point A$(1, -8)$.

(*a*) Find the equation of the tangent at A and hence find the coordinates of B. **(8)**

(*b*) Find the area of the shaded part which represents the land bounded by the river and the road. **(3)**



Part	Marks	Level	Calc.	Content	Answer	U2 OC2
(*a*)	5	C	CN	C4, G3, A23		1996 P2 Q8
(*a*)	3	A/B	CN	C4, G3, A23		
(*b*)	3	A/B	CN	C17		

(a)

•1 *strat*: $\frac{dy}{dx} = \ldots$

•2 $\frac{dy}{dx} = 3x^2 - 2x - 6$

•3 $m_{tgt} = -5$

•4 $y + 8 = -5(x - 1)$

•5 *strat*: attempt to simplify and equate y's

•6 $x^3 - x^2 - x + 1 = 0$

•7 *strat*: *e.g.* try to factorise

•8 $B = (-1, 2)$

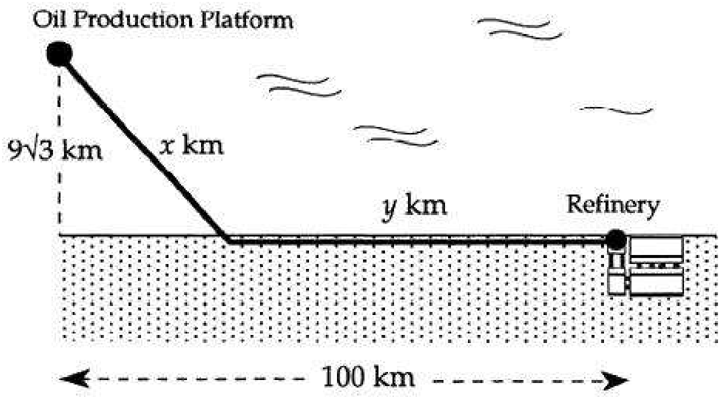
(b)

•9 $\int (x^3 - x^2 - 6x - 2) - (-5x - 3)\ dx$

•10 $\left[\frac{1}{4}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + x\right]$

•11 $1\frac{1}{3}$

[SQA] 19. An oil production platform, $9\sqrt{3}$ km offshore, is to be connected by a pipeline to a refinery on shore, 100 km down the coast from the platform as shown in the diagram.



The length of underwater pipeline is x km and the length of pipeline on land is y km. It costs £2 million to lay each kilometre of pipeline underwater and £1 million to lay each kilometre of pipeline on land.

(*a*) Show that the total cost of this pipeline is £$C(x)$ million where

$$C(x) = 2x + 100 - \left(x^2 - 243\right)^{\frac{1}{2}}.$$ **(3)**

(*b*) Show that $x = 18$ gives a minimum cost for this pipeline.
Find this minimum cost and the corresponding total length of the pipeline. **(7)**

Part	Marks	Level	Calc.	Content	Answer	U3 OC2
(*a*)	1	C	NC	A6		1993 P2 Q11
(*a*)	2	A/B	NC	A6		
(*b*)	1	C	NC	C11, C21		
(*b*)	6	A/B	NC	C11, C21		

(*a*) •[1] $C = 2x + y$

•[2] $\sqrt{x^2 - \left(9\sqrt{3}\right)^2}$

•[3] for completing proof

(*b*) •[4] knowing to differentiate

•[5] $\frac{1}{2}\left(x^2 - 243\right)^{-\frac{1}{2}}$

•[6] $\times\ 2x$

•[7] $C'(18) = 0$

•[8] justification of minimum e.g. nature table

•[9] $C = 127$

•[10] $x + y = 109$

	18^-	18	18^+
$C'(x)$	$-$	0	$+$
	↘	—	↗
		minimum	

[SQA] 20. If $f(x) = \cos^2 x - \dfrac{2}{3x^2}$, find $f'(x)$. **4**

Part	Marks	Level	Calc.	Content	Answer	U3 OC2
	2	C	NC	C21, C1		1990 P1 Q19
	2	A/B	NC	C21, C1		

•1 $-\frac{2}{3}x^{-2}$
•2 $2\cos x$
•3 $\times(-\sin x)$
•4 $\frac{4}{3}x^{-3}$

[SQA] 21. Differentiate $4\sqrt{x} + 3\cos 2x$ with respect to x. **4**

Part	Marks	Level	Calc.	Content	Answer	U3 OC2
	2	C	NC	C21, C1		1993 P1 Q9
	2	A/B	NC	C21, C1		

•1 $4x^{\frac{1}{2}}$
•2 $2x^{-\frac{1}{2}}$
•3 $-\sin 2x$
•4 $\times 2$

[END OF QUESTIONS]